STAT 517:Sufficiency

Properties of a Sufficient Statistic. Completeness and Uniqueness

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Some remarks

- A sufficient statistic is never unique in any sense
- If $Y_1 = u_1(X_1, \ldots, X_n)$ is a sufficient statistic and $g(x)$ is a one-to-one function, $Y_2 = g(Y_1)$ is also a sufficient statistic
- Let Y_2 be an unbiased statistic of θ
- We have
$$\theta = \mathbb{E}\left(Y_2\right) = \mathbb{E}\left[Y_2|y_1\right] = \phi(y_1)$$
- Note that
$$Var\ Y_2 \geq Var\left[\phi(Y_1)\right]$$

Rao-Blackwell Theorem

- Thus, $\mathbb{E}\left(Y_2|y_1\right) = \phi(y_1)$ defines a statistic $\phi(Y_1)$
- $\phi(Y_1)$ is an unbiased estimator of θ with the variance not exceeding that of Y_2
- In other words, when looking for an MVUE, we can just stick to functions of a sufficient statistic
- Note that it is *not necessary* to start with some unbiased estimator Y_2

Example

- Let $X_1, \ldots, X_n \sim b(1, \theta)$
- X_1 is an unbiased estimator of θ and $T = \sum_{i=1}^n X_i$
- Thus,

$$\mathbb{E}\left(X_1 | \sum_{i=1}^n X_i = t\right) = \frac{P_\theta(X_1 = 1, \sum_{i=1}^n X_i = t)}{P_\theta(\sum_{i=1}^n X_i = t)}$$

$$= \frac{\theta\binom{n-1}{t-1}\theta^{t-1}(1-\theta)^{n-1-(t-1)}}{\binom{n}{t}\theta^t(1-\theta)^{n-t}} = \frac{t}{n}$$

- Thus, $\bar{X} = \frac{T}{n}$ is an unbiased estimator of θ with a smaller variance

Remark

- Note that for any sample of iid RV's $X_1, \ldots, X_n \sim f(x;\theta)$ $\sum_{i=1}^n X_i$ is symmetric in $X_1, \ldots, X_n$
- Thus, $\mathbb{E}\left(X_i | \sum_{i=1}^n X_i = t\right)$ are the same for all i's
- Therefore,

$$\mathbb{E}\left(X_1 | \sum_{i=1}^n X_i = t\right) = \frac{1}{n} \sum_{j=1}^n \mathbb{E}\left(X_j | \sum_{i=1}^n X_i = t\right) = \frac{t}{n}$$

MLE and sufficient statistics

- Note that if a unique MLE $\hat{\theta}$ exists, it must be a function of a sufficient statistic $Y_1 = u_1(X_1, \ldots, X_n)$.
- This suggests the following algorithm:
  1. Find a sufficient statistic
  2. Obtain a (unique) MLE
  3. If necessary, do the bias correction of such an MLE

Example

- Let $X_1, \dots, X_n \sim Exp(\theta)$ with $\mathbb{E}\,X = \frac{1}{\theta}$
- Confirm that the MLE is

$$Y_2 = \frac{1}{\bar{X}}$$

- By factorization theorem, $Y_1 = \bar{X}$ is a sufficient statistic
- Each $X_i \sim \Gamma\left(1, \frac{1}{\theta}\right)$ and $Y_1 \sim \Gamma\left(n, \frac{1}{\theta}\right)$
- Check that $\mathbb{E}\left(Y_2\right) = \theta\frac{n}{n-1}$
- Due to the above, $\frac{(n-1)Y_2}{n} = \frac{n-1}{\sum_{i=1}^{n} X_i}$ is an MVUE of θ

Complete family of pdf(pmf)

- Take $X_1, \dots, X_n \sim Pois(\theta)$
- Recall that $Y_1 = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ and define $u(Y_1)$ s.t. $\mathbb{E}\left[u(Y_1)\right] = 0$ for every $\theta > 0$
- This immediately implies that

$$u(0) = u(1) = \cdots = 0$$

- In general, if $\mathbb{E}\left[u(Z)\right] = 0$ for any $\theta \in \Omega$ implies that $u(z) = 0$ except on a set of points of probability zero, $\{h(z;\theta) : \theta \in \Omega\}$ is a **complete family of pdf/pmfs**

Example

- Again, let $Z \sim Exp(\theta)$ s.t. $\mathbb{E}\, Z = \theta$
- $\mathbb{E}\,[u(Z)] = 0$ implies that

$$\frac{1}{\theta}\int_0^\infty u(z)e^{-z/\theta}\,dz = 0$$

- The Laplace transform being zero means that $u(z) = 0$ everywhere except a set of points of probability zero
- Therefore, $h(z;\theta) = \frac{1}{\theta}e^{-z/\theta}$ for any $0 < z < \infty$ is complete

A statistic that is sufficient but not complete

- Take $X_1, \ldots, X_n \sim N(\mu, a^2\mu^2)$ with known $a > 0$
- We know that $Y = \left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right)$ is a joint sufficient statistic for μ
- Note that $\mathbb{E}\left(\sum_{i=1}^n X_i^2\right) = n(\mu^2 + a^2\mu^2)$ and, therefore,

$$\mathbb{E}\left\{\frac{n+a^2}{1+a^2}\sum_{i=1}^n X_i^2 - \left(\sum_{i=1}^n X_i\right)^2\right\} = 0$$

and so Y is not complete

Completeness and Lehmann-Scheffe Theorem

- Let $Y_1 = u_1(Y_1, \dots, Y_n)$ be a sufficient statistic for θ
- Let the family of pdfs $f_{Y_1}(y_1; \theta)$ be complete
- Then, if $\phi(Y_1)$ is an unbiased estimator of θ, it is the unique MVUE of θ
- This statement is known as the **Lehmann-Scheffe Theorem**
- Instead of saying "Y_1 is a sufficient statistic and the family of pdfs $f_{Y_1}(y_1; \theta)$ is complete", it is usually simply said that Y_1 is a **complete sufficient statistic**

Example

- Let $X_1, \ldots, X_n \sim Unif[0, \theta]$
- Thus, $f(x;\theta) = \frac{1}{\theta}$ for $0 < x < \theta$ and 0 otherwise
- Recall $Y_n = \max\{X_1, \ldots, X_n\}$ is a sufficient statistic for θ
- Note that the pdf of Y_n is $g(y_n;\theta) = \frac{ny_n^{n-1}}{\theta^n}$ for $0 < y_n < \theta$
- If, for some $u(t)$, $\mathbb{E}[u(Y_n)] = 0$, we have

$$\int_0^\theta u(t)\frac{nt^{n-1}}{\theta^n}\,dt = 0$$

Example

- Since $\theta > 0$, dividing by θ^n and differentiating w.r.t. θ, we have
$$u(\theta)\theta^{n-1} = 0$$
- Thus, $u(\theta) = 0$ for all $\theta > 0$ and Y_n is a complete sufficient statistic for θ
- Additionally, $\mathbb{E}\, Y_n = \frac{n}{n+1}\theta$ and so the MVUE of θ is $\frac{n+1}{n} Y_n$

Example

- Let $Y_1, \dots, Y_n \sim Pois(\lambda)$ be iid annual numbers of earthquakes in a region
- Estimate the probability $p = P(Y \geq 1) = 1 - e^{-\lambda}$ - the probability of an earthquake in the next year
- Let $T = 1$ if $Y \geq 1$ and 0 otherwise; then, T is an unbiased estimator for p
- $W = \sum_{i=1}^{n} Y_i$ is a sufficient statistic for p
- Accordingly,

$$T|W \sim b\left(1, P\left(Y \geq 1 | \sum_{i=1}^{n} Y_i = W\right)\right)$$

Example

- Define

$$
\begin{aligned}
T_1 = \mathbb{E}\left(T|W\right) &= 1 - P\left(Y_1 = 0 | \sum_{i=1}^{n} Y_i = W\right) \\
&= 1 - \frac{P\left(Y_1 = 0, \sum_{i=1}^{n} Y_i = W\right)}{P\left(\sum_{i=1}^{n} Y_i = W\right)} \\
&= 1 - \frac{P(Y_1 = 0) P\left(\sum_{i=2}^{n} Y_i = W\right)}{P\left(\sum_{i=1}^{n} Y_i = W\right)} \\
&= 1 - \left(\frac{n-1}{n}\right)^{W} = 1 - \left(1 - \frac{1}{n}\right)^{\sum_{i=1}^{n} Y_i}
\end{aligned}
$$

- Since W is a complete sufficient statistic for λ, the estimator T_1 is an MVUE of λ