STAT 517: Sufficiency
Properties of a Sufficient Statistic. Completeness and Uniqueness

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Some remarks

- A sufficient statistic is never unique in any sense
- If $Y_1 = u_1(X_1, \ldots, X_n)$ is a sufficient statistic and $g(x)$ is a one-to-one function, $Y_2 = g(Y_1)$ is also a sufficient statistic
- Let $Y_2$ be an unbiased statistic of $\theta$
- We have
  $$\theta = \mathbb{E}(Y_2) = \mathbb{E}[Y_2|y_1] = \phi(y_1)$$
- Note that
  $$\text{Var} \ Y_2 \geq \text{Var} [\phi(Y_1)]$$
Thus, $\mathbb{E}(Y_2|y_1) = \phi(y_1)$ defines a statistic $\phi(Y_1)$

$\phi(Y_1)$ is an unbiased estimator of $\theta$ with the variance not exceeding that of $Y_2$

In other words, when looking for an MVUE, we can just stick to functions of a sufficient statistic

Note that it is not necessary to start with some unbiased estimator $Y_2$
Example

- Let $X_1, \ldots, X_n \sim b(1, \theta)$
- $X_1$ is an unbiased estimator of $\theta$ and $T = \sum_{i=1}^{n} X_i$
- Thus,

$$
\mathbb{E} \left( X_1 \mid \sum_{i=1}^{n} X_i = t \right) = \frac{P_{\theta}(X_1 = 1, \sum_{i=1}^{n} X_i = t)}{P_{\theta}(\sum_{i=1}^{n} X_i = t)} \\
= \frac{\theta^{(n-1)} \theta^{t-1} (1 - \theta)^{n-1-(t-1)}}{\binom{n}{t} \theta^t (1 - \theta)^{n-t}} = \frac{t}{n}
$$

- Thus, $\bar{X} = \frac{T}{n}$ is an unbiased estimator of $\theta$ with a smaller variance
Remark

- Note that for any sample of iid RV's $X_1, \ldots, X_n \sim f(x; \theta)$, $\sum_{i=1}^{n} X_i$ is symmetric in $X_1, \ldots, X_n$.
- Thus, $\mathbb{E} \left( X_i \mid \sum_{i=1}^{n} X_i = t \right)$ are the same for all $i$'s.
- Therefore,

$$
\mathbb{E} \left( X_1 \mid \sum_{i=1}^{n} X_i = t \right) = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left( X_j \mid \sum_{i=1}^{n} X_i = t \right) = \frac{t}{n}
$$
Note that if a unique MLE $\hat{\theta}$ exists, it must be a function of a sufficient statistic $Y_1 = u_1(X_1, \ldots, X_n)$.

This suggests the following algorithm:

1. Find a sufficient statistic
2. Obtain a (unique) MLE
3. If necessary, do the bias correction of such an MLE
Example

Let $X_1, \ldots, X_n \sim \text{Exp}(\theta)$ with $\mathbb{E} X = \frac{1}{\theta}$

Confirm that the MLE is

$$Y_2 = \frac{1}{\bar{X}}$$

By factorization theorem, $Y_1 = \bar{X}$ is a sufficient statistic

Each $X_i \sim \Gamma \left(1, \frac{1}{\theta}\right)$ and $Y_1 \sim \Gamma \left(n, \frac{1}{\theta}\right)$

Check that $\mathbb{E} (Y_2) = \theta \frac{n}{n-1}$

Due to the above, $\frac{(n-1)Y_2}{n} = \frac{n-1}{\sum_{i=1}^{n} X_i}$ is an MVUE of $\theta$
Complete family of pdf(pmfs)

- Take $X_1, \ldots, X_n \sim \text{Pois}(\theta)$
- Recall that $Y_1 = \sum_{i=1}^{n} X_i$ is a sufficient statistic for $\theta$ and define $u(Y_1)$ s.t. $\mathbb{E}[u(Y_1)] = 0$ for every $\theta > 0$
- This immediately implies that
  \[
  u(0) = u(1) = \cdots = 0
  \]
- In general, if $\mathbb{E}[u(Z)] = 0$ for any $\theta \in \Omega$ implies that $u(z) = 0$ except on a set of points of probability zero, $\{h(z; \theta) : \theta \in \Omega\}$ is a **complete family of pdf/pmf**
Example

- Again, let $Z \sim \text{Exp}(\theta)$ s.t. $\mathbb{E} Z = \theta$
- $\mathbb{E}[u(Z)] = 0$ implies that
  \[
  \frac{1}{\theta} \int_{0}^{\infty} u(z) e^{-z/\theta} \, dz = 0
  \]
- The Laplace transform being zero means that $u(z) = 0$ everywhere except a set of points of probability zero
- Therefore, $h(z; \theta) = \frac{1}{\theta} e^{-z/\theta}$ for any $0 < z < \infty$ is complete
A statistic that is sufficient but not complete

- Take $X_1, \ldots, X_n \sim N(\mu, a^2 \mu^2)$ with known $a > 0$
- We know that $Y = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ is a joint sufficient statistic for $\mu$
- Note that $\mathbb{E} \left( \sum_{i=1}^n X_i^2 \right) = n(\mu^2 + a^2 \mu^2)$ and, therefore,

$$
\mathbb{E} \left\{ \frac{n + a^2}{1 + a^2} \sum_{i=1}^n X_i^2 - \left( \sum_{i=1}^n X_i \right)^2 \right\} = 0
$$

and so $Y$ is not complete
Completeness and Lehmann-Scheffe Theorem

- Let $Y_1 = u_1(Y_1, \ldots, Y_n)$ be a sufficient statistic for $\theta$
- Let the family of pdfs $f_{Y_1}(y_1; \theta)$ be complete
- Then, if $\phi(Y_1)$ is an unbiased estimator of $\theta$, it is the unique MVUE of $\theta$
- This statement is known as the Lehmann-Scheffe Theorem
- Instead of saying “$Y_1$ is a sufficient statistic and the family of pdfs $f_{Y_1}(y_1; \theta)$ is complete”, it is usually simply said that $Y_1$ is a complete sufficient statistic
Example

- Let \( X_1, \ldots, X_n \sim Unif[0, \theta] \)
- Thus, \( f(x; \theta) = \frac{1}{\theta} \) for \( 0 < x < \theta \) and 0 otherwise
- Recall \( Y_n = \max\{X_1, \ldots, X_n\} \) is a sufficient statistic for \( \theta \)
- Note that the pdf of \( Y_n \) is \( g(y_n; \theta) = \frac{ny_n^{n-1}}{\theta^n} \) for \( 0 < y_n < \theta \)
- If, for some \( u(t) \), \( \mathbb{E}[u(Y_n)] = 0 \), we have

\[
\int_0^\theta u(t) \frac{nt^{n-1}}{\theta^n} \, dt = 0
\]
Example

- Since $\theta > 0$, dividing by $\theta^n$ and differentiating w.r.t. $\theta$, we have
  \[ u(\theta)\theta^{n-1} = 0 \]

- Thus, $u(\theta) = 0$ for all $\theta > 0$ and $Y_n$ is a complete sufficient statistic for $\theta$

- Additionally, $\mathbb{E} Y_n = \frac{n}{n+1} \theta$ and so the MVUE of $\theta$ is $\frac{n+1}{n} Y_n$
Example

- Let $Y_1, \ldots, Y_n \sim \text{Pois}(\lambda)$ be iid annual numbers of earthquakes in a region
- Estimate the probability $p = P(Y \geq 1) = 1 - e^{-\lambda}$ - the probability of an earthquake in the next year
- Let $T = 1$ if $Y \geq 1$ and 0 otherwise; then, $T$ is an unbiased estimator for $p$
- $W = \sum_{i=1}^{n} Y_i$ is a sufficient statistic for $p$
- Accordingly,

$$T|W \sim b\left(1, P\left(Y \geq 1 | \sum_{i=1}^{n} Y_i = W\right)\right)$$
Example

Define

\[ T_1 = \mathbb{E} (T \mid W) = 1 - P \left( Y_1 = 0 \mid \sum_{i=1}^{n} Y_i = W \right) \]

\[ = 1 - \frac{P(Y_1 = 0, \sum_{i=1}^{n} Y_i = W)}{P(\sum_{i=1}^{n} Y_i = W)} \]

\[ = 1 - \frac{P(Y_1 = 0)P(\sum_{i=2}^{n} Y_i = W)}{P(\sum_{i=1}^{n} Y_i = W)} \]

\[ = 1 - \left( \frac{n-1}{n} \right)^W = 1 - \left( 1 - \frac{1}{n} \right)^{\sum_{i=1}^{n} Y_i} \]

Since \( W \) is a complete sufficient statistic for \( \lambda \), the estimator \( T_1 \) is an MVUE of \( \lambda \)