STAT 517:Sufficiency

Measures of Quality of Estimators

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Minimum variance unbiased estimators(MVUE)

- So far, we considered consistency and unbiasedness
- Recall that MLE's are not always unbiased although they tend to be asymptotically unbiased under a set of regularity conditions
- The model :$X_1, \dots, X_n \sim f(x; \theta)$ for $\theta \in \Omega$
- For a given $n > 0$, $Y = u(X_1, \dots, X_n)$ is a **minimum variance unbiased estimator (MVUE)** of θ if Y is unbiased and the variance of Y is less than or equal to the variance of *every other unbiased estimator* of θ

Example

- Let $X_1, \ldots, X_n \sim N(\theta, \sigma^2)$
- Since $\bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right)$ $\bar{X}$ is an unbiased estimator of θ...but so it X_1
- Clearly, $Var\, \bar{X} < Var\, X_1$ for any $n > 1$...but $\bar{X}$ is not a minimum variance unbiased estimator!!

Decision rules and loss functions

- Any function $\delta(Y)$ is a **decision function** or a **decision rule**
- A specific values $\delta(y)$ is a **decision**
- To measure how different $\delta(y)$ is from θ, use the **loss function** $L(\theta, \delta(y))$
- The loss function is random...better to use the **risk function**

$$R(\theta, \delta) = \mathbb{E}\left\{L(\theta, \delta(y))\right\} = \int_{-\infty}^{\infty} L(\theta, \delta(y)) f_Y(y; \theta)\, dy$$

- Problem: uniform minimization of risk function over *all* possible θ may be impossible

Example

- Model: $X_1, \ldots, X_n \sim N(\theta, 1)$ and let $Y = \bar{X}$
- Choose the **mean squared error loss** $L(\theta, \delta(y)) = [\theta - \delta(y)]^2$
- How to choose between $\delta(y) = y$ and $\delta_2(y) = 0$? The risk functions are

$$R(\theta, \delta_1) = \mathbb{E}\left[(\theta - Y)\right]^2 = \frac{1}{n}$$

and

$$R(\theta, \delta_1) = \mathbb{E}\left[(\theta - 0)^2\right] = \theta^2$$

Example

- If $\theta = 0$, then the 2nd choice is better...but if θ is far from zero, $\delta_2 = 0$ is a bad choice!
- If considering only $\delta(y) : \mathbb{E}[(\delta(Y)] = 0$, then δ_2 is not allowed
- Under the latter restriction, we are looking for an MVUE which is, actually, $\bar{X}$ (to be shown later)
- Yet another possible alternative is to use the **minimax criterion**: $\delta_0(y)$ is a **minimax decision function** if, for all $\theta \in \Theta$,
$$\max_{\theta} R[\theta, \delta_0(y)] \leq \max_{\theta} R[\theta, \delta(y)]$$
for any other decision function $\delta(y)$
- Note $R(\theta, \delta_2) = \theta^2$ which is unbounded if $-\infty < \theta < \infty$ and so has to be excluded according to the minimax criterion
- Actually (not proven by us here..) δ_1 is the best choice according to the minimax decision function

Remarks

- Another possibility is simply to define $\delta(X_1, \ldots, X_n)$ without using a statistic Y - will not do it here
- Besides the squared error loss function, can also consider e.g. the **absolute-error loss function**

$$L(\theta, \delta) = \begin{cases} 0 & |\theta - \delta| \leq a \\ b & |\theta - \delta| > a \end{cases}$$

for some $a, b > 0$

- This is sometimes called the *goalpost loss function*

Likelihood principle

- A scientist A observes 10 independent trials with prob. of success $0 < \theta < 1$ and has only 1 success
- A scientist B observes all trials until the first success which happens to be the 10th
- First model: $Y \sim B(10, \theta)$ with observed number of successes $y = 1$; the second model is $g(z) = (1 - \theta)^{z-1}\theta$ with $z = 10$
- A sensible estimate in both cases is the relative frequency: $\hat{\theta} = \frac{y}{n} = \frac{1}{z} = \frac{1}{10}$
- $\hat{\theta}$ is unbiased in the first case but not in the second!

Likelihood principle

- The first likelihood is

$$L_1(\theta) = \binom{10}{y}\theta^y(1-\theta)^{10-y}$$

- The second likelihood is

$$L_2(\theta) = (1-\theta)^{z-1}\theta$$

- When $z = 10$ and $y = 1$, both are proportional to $(1-\theta)^9\theta$...and both give the same answer $\hat{\theta} = \frac{1}{10}$
- To a true believer in the **likelihood principle** the fact that one of them is unbiased does not matter!