STAT 517:Maximum Likelihood Methods

Lecture 9: Multiparameter case: Testing

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Framework

- $X_1, \ldots, X_n \sim f(x; \boldsymbol{\theta})$ for $\boldsymbol{\theta} \subset \Omega \in \mathbb{R}^p$
- The hypotheses of interest are: $H_0 : \boldsymbol{\theta} \in \omega$ vs. $H_1 : \boldsymbol{\theta} \in \Omega \cap \omega^c$
- In the above, $\omega \in \Omega$ is defined in terms of $0 < q \leq p$ independent constraints of the form $g_1(\boldsymbol{\theta}) = a_1, \ldots,$ $g_q(\boldsymbol{\theta}) = a_q$. Moreover, the functions $g_1, \ldots, g_q$ must be continuously differentiable
- For a $p - q$-dimensional space ω, we define the likelihood ratio

$$\Lambda = \frac{\max_{\boldsymbol{\theta} \in \omega} L(\boldsymbol{\theta})}{\max_{\boldsymbol{\theta} \in \Omega} L(\boldsymbol{\theta})}$$

- As before, large (close to 1) values of Λ suggest that H_0 is true while small ones are the evidence in favor of H_1
- For the significance level α the decision rule is to reject H_0 if $\Lambda \leq c$ where c is the solution of $\alpha = \max_{\boldsymbol{\theta} \in \omega} P_{\boldsymbol{\theta}}[\Lambda \leq c]$

Likelihood ratio test

- Let $\hat{\boldsymbol{\theta}}$ be the MLE estimator of $\boldsymbol{\theta}$ over Ω and $\hat{\boldsymbol{\theta}}_0$ the MLE over ω
- Define $L(\hat{\Omega}) = L(\hat{\boldsymbol{\theta}})$ and $L(\hat{\omega}) = L(\hat{\boldsymbol{\theta}}_0)$
- The **likelihood ratio test (LRT)** statistic is

$$\Lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

Example

- $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$
- Test $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$; the full parameter space is $\Omega = \{(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0\}$ and the reduced is $\omega = \{(\mu_0, \sigma^2) : \sigma^2 > 0\}$
- The MLE's are $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

A toy example

- Possible country of origin: UK or France
- Choice of alcohol: beer, brandy/cognac, whisky, wine
- The distribution is

State	beer	brandy	whisky	wine
France	10%	20%	10%	60%
UK	50%	10%	20%	20%

- H_0 : France vs H_1 : UK with $\alpha = 0.25$
- There are four parameters $\theta_1 = P(beer)$, $\theta_2 = P(brandy)$ etc.

A toy example

- Thus, the hypotheses are:
  $H_0 : \theta_1 = 0.1, \theta_2 = 0.2, \theta_3 = 0.1, \theta_4 = 0.6$ vs.
  $H_1 : \theta_1 = 0.5, \theta_2 = 0.1, \theta_3 = 0.2, \theta_4 = 0.2$
- Possible values of the likelihood ratio statistic are 5 for beer, 0.5 for brandy, 2 for whiskey, and $1/3$ for wine

Example

- Confirm that the LRT statistic is

$$\Lambda = \left(\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sum_{i=1}^n (X_i - \mu_0)^2} \right)^{n/2}$$

- $\Lambda \leq c$ is equivalent to $\Lambda^{-n/2} \geq c'$; so we have

$$1 + \frac{n(\bar{X} - \mu_0)^2}{\sum_{i=1}^n (X_i - \bar{X})^2} \geq c'$$

or

$$\left\{ \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2/(n-1)}} \right\}^2 = T^2 \geq c'' = (c' - 1)(n-1)$$

- The usual decision rule is, then, reject H_0 if $|T| \geq c^*$ where $\alpha = P_{H_0}[|T| \geq c^*]$; this is the two-sided version of the t-test