STAT 516

Multivariate case: multinomial distribution

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Definition

- Let $X_1, X_2, \ldots, X_n$ be the discrete random variables defined on a common sample space Ω.
- Each X_i takes values in some countable set $\aleph_i$
- The joint pmf of $(X_1, X_2, \ldots, X_n)$ is defined as

$$p(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$$

whenever $x_i \in \aleph_i$ and 0 otherwise

Requirements of joint pmf

1. $$p(x_1, \ldots, x_n) \geq 0$$

   for any $x_1, \ldots, x_n \in R$

2. $$\sum_{x_1 \in \aleph_1, \ldots, x_n \in \aleph_n} p(x_1, \ldots, x_n) = 1$$

Multinomial distribution

- Let n balls be distributed to k cells independently
- Each ball has the probability p_i of being dropped into the ith cell
- We consider X_i, $i = 1, \ldots, k$ as the number of balls that gets dropped into the ith cell
- Their joint pmf with parameters n, $p_1, \ldots, p_k$ is

$$P(X_1 = x_1, \ldots, X_k = x_k) = \frac{n!}{x_1! \ldots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

  where $x_i \geq 0$, $\sum_{i=1}^k x_i = n$, $p_i \geq 0$ and $\sum_{i=1}^k p_i = 1$
- We write $(X_1, \ldots, X_k) \sim Mult(n, p_1, \ldots, p_k)$ - a random vector with a multinomial distribution

Example I

1. Suppose a fair die is rolled 30 times
2. We have $(X_1, \ldots, X_6) \sim Mult(30, p_1, \ldots, p_6)$ where each $p_i = \frac{1}{6}$
3. The probability

$$P(X_1 = 5, \ldots, X_6 = 5) = \frac{30!}{(5!)^6} \left(\frac{1}{6}\right)^5 \cdots \left(\frac{1}{6}\right)^5$$
$$= 0.0004$$

Example I

1. Note that each of the thirty rolls is either six or not
2. The probability of getting 6 on each roll is $\frac{1}{6}$ and the rolls are independent
3. Since $X_6 \sim Bin(30, 1/6)$,

$$P(X_6 \geq 5) = 1 - P(X_6 \leq 4) = 1 - \sum_{x=0}^{4} \binom{30}{x} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{30-x}$$

$$= .5757$$

Basic properties of the multinomial distribution

- Let $(X_1, \ldots, X_k) \sim Mult(n, p_1, \ldots, p_k)$. Then
  1. $E(X_i) = np_i$; $Var(X_i) = np_i(1 - p_i)$
  2. For any i, $X_i \sim Bin(n, p_i)$
  3. $Cov(X_i, X_j) = -np_i p_j$ for any $i \neq j$
  4. $\rho_{X_i, X_j} = -\sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}}$ for any $i \neq j$
  5. For any m such that $1 \leq m \leq k$ the conditional distribution

     $$(X_1, \ldots, X_m) | (X_{m+1} + X_{m+2} + \cdots + X_k) = s \sim Mult(n-s, \theta_1, \ldots, \theta_m)$$

     where $\theta_i = \frac{p_i}{p_1 + \cdots + p_m}$

Basic properties of the multinomial distribution

- (1) Denote W_{ir} as the indicator of the event that rth ball lands in the ith cell
- For given i, variables W_{ir} are independent; thus, $X_i = \sum_{r=1}^{n} W_{ir}$
- Therefore, $E(X_i) = E(\sum_{r=1}^{n} W_{ir}) = np_i$ and $Var(X_i) = Var(\sum_{r=1}^{n} W_{ir}) = np_i(1 - p_i)$
- (2) follows from the definition of a multinomial experiment