STAT 516: Multivariate Distributions

Expectation of Functions

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Definition

- Let $(X_1, X_2, \ldots, X_n)$ have a joint density function $f(x_1, x_2, \ldots, x_n)$
- Let $g(x_1, x_2, \ldots, x_n)$ be a real-valued function of $x_1, x_2, \ldots, x_n$.
- The expectation of $g(X_1, X_2, \ldots, X_n)$ exists if

$$\int_{R^n} |g(x_1, x_2, \ldots, x_n)| f(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n < \infty$$

- Then, the expected value of $g(X_1, X_2, \ldots, X_n)$ is

$$E[g(X_1, \ldots, X_n)] = \int_{R^n} g(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

Remark

- The expectation of each individual X_i can be obtained by either
  1. interpreting X_i as a function of the full vector $(X_1, \dots, X_i)$

     $$E(X_i) = \int_{R^n} x_i f(x_1, \dots, x_n) \, dx_1 \dots dx_n$$

     or by
  2. simply using the marginal density function $f_i(x)$ of X_i

     $$E(X_i) = \int_{-\infty}^{\infty} x f_i(x) \, dx$$

- Also, all of the earlier established expectation properties are still applicable. Thus, linearity implies that

$$E[ag(X_1, \dots, X_n) + bh(X_1, \dots, X_n)]$$
$$= aE[g(X_1, \dots, X_n)] + bE[h(X_1, \dots, X_n)]$$

Examples

- Bivariate Uniform on $[0,1]^2$
- The expected distance between the two coordinates is

$$
\begin{aligned}
E(|X-Y|) &= \int_0^1 \int_0^1 |x-y| dxdy \\
&= \int_0^1 \left[\int_0^y (y-x)dx + \int_y^1 (x-y)dx \right] dy \\
&= \int_0^1 \left[\left(y^2 - \frac{y^2}{2} \right) + \left(\frac{1-y^2}{2} - y(1-y) \right) \right] dy \\
&= \int_0^1 \left[\frac{1}{2} - y + y^2 \right] dy \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{3} = \frac{1}{3}
\end{aligned}
$$

Example: uniform in a triangle

- The uniform density on a triangle $x, y \geq 0$ and $x + y \leq 1$ is $f(x, y) = 2$ - recall a previous example
- The marginal density of X is $f(x) = 2(1 - x)$ for $0 \leq x \leq 1$. Thus,

$$E(X) = \int_0^1 2x(1-x)dx = \frac{1}{3}$$

- Due to symmetry of X and Y the marginal density of Y is the same...as well as its marginal expectation
- Next,

$$E(X^2) = \int_0^1 2x^2(1-x)dx = \frac{1}{6}$$

and so $Var(X) = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$... Once again, $Var(Y)$ is the same

Example: uniform in a triangle

- Also,

$$E(XY) = 2\int_0^1 \int_0^{1-y} xy\, dxdy = \frac{1}{12}$$

- Therefore,

$$Cov(X, Y) = E(XY) - E(X)E(Y) = -\frac{1}{36}$$

and

$$\rho_{X,Y} = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-\frac{1}{36}}{\frac{1}{18}} = -\frac{1}{2}$$