STAT 516

Bivariate Joint Distributions and Expectations of Functions

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Motivation

- Consider the experiment of tossing ξ of tossing a fair coin three times.
- Let X be the number of heads among the first two tosses and Y the number of heads among the last two tosses
- We cannot consider X and Y as two independent $\text{Bin}(2, 0.5)$ random variables; indeed, if $X = 2$, then $Y \geq 1$

Motivation

- The combination of all possible values of (X, Y) are: $(0,0),(0,1),(1,0),(0,2),(2,0),(1,1),(1,2),(2,1)$.
- The joint probability mass function can be described as

	Y		
X	0	1	2
0	$\frac{1}{8}$	$\frac{1}{8}$	0
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$

Definition

- Let (X, Y) be two discrete random variables with sets of values $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ on a common sample space Ω.
- The joint pmf of (X, Y) is the function $p(x_i, y_j) = P(X = x_i, Y = y_j)$ for $i, j \geq 1$ and $p(x, y) = 0$ on any other point (x, y)
- The requirements:
  1. $p(x, y) \geq 0 \forall (x, y)$
  2. $\sum_i \sum_j p(x_i, y_j) = 1$

Definition

- Let $p(x, y)$ be the joint pmf of (X, Y). The marginal pmf of a function $Z = g(X, Y)$ is

$$p_Z(z) = \sum_{(x,y):g(x,y)=z} p(x, y)$$

- In particular,

$$p_X(x) = \sum_y p(x, y)$$

and

$$p_Y(y) = \sum_x p(x, y)$$

- Moreover, for any event A

$$P(A) = \sum_{(x,y)\in A} p(x, y)$$

Example I

- Let X be the number of aces in the hands of Player 1 and Y the number of aces in the hands of Player 2 in a bridge game
- Then, $X, Y \geq 0$ and $X + Y \leq 4$
- If Player 1 gets x aces, Player 2 can get y aces in $\binom{4-x}{y}$ ways
- Player 1 also has to get $13 - x$ non-ace cards and Player 2 $13 - y$ non-ace cards

Example I

- Thus, the pmf is

$$p(x,y) = \frac{\binom{4}{x}\binom{48}{13-x}\binom{4-x}{y}\binom{35+x}{13-y}}{\binom{52}{13}\binom{39}{13}}$$

where $x \geq 0$, $y \geq 0$, and $x + y \leq 4$

- For example, $p(1,0) = .1249$, $p(1,1) = .0974$, $p(1,2) = 0.974$, $p(1,3) = .0137$, and $p(1,4) = 0$
- Summing, we get

$$P(X = 1) = \sum_y p(1,y) = .4389$$

which can also be obtained from the direct formula for the pmf of X :

$$P(X = 1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} = .4389$$

Example I

- Let $A: X + Y = 2$ be the event of interest
- $$P(A) = \sum_{(x,y)\in A} p(x,y) = p(0,2) + p(1,1) + p(2,0) = .3901$$
- Note that this probability cannot be computed using only marginal distributions of X and Y

Example II

- Let
$$p(x, y) = c(x + y)$$
for $1 \le x, y \le n$
- To find the constant c we need to equate
$$\sum_{x=1}^{n} \sum_{y=1}^{n} p(x, y) = 1$$
- This results in
$$c \sum_{x=1}^{n} \left[nx + \frac{n(n+1)}{2} \right] = 1$$
and, finally,
$$c = \frac{1}{n^2(n+1)}$$

Example II

- Note that this joint pmf is symmetric because $x + y = y + x$...and, so, $p(x, y) = p(y, x)$
- That implies, in turn, that X and Y have the same marginal pmf
- E.g.

$$p_X(x) = \sum_{y=1}^{n} p(x, y) = \frac{1}{n^2(n+1)} \sum_{y=1}^{n} (x + y)$$

$$= \frac{x}{n(n+1)} + \frac{1}{2n}$$

for any $1 \leq x \leq n$

Example II

- How to compute $P(X > Y)$
- By symmetry, $P(X > Y) = P(Y > X)$ and so

$$P(X > Y) + P(Y > X) + P(X = Y) = 1$$

- Denote $P(X < Y) = P(Y < X) = p$; thus, $2p + P(X = Y) = 1$ and $p = \frac{1-P(X=Y)}{2}$
- Now,

$$P(X = Y) = \sum_{x=1}^{n} p(x, x) = c \times \sum_{x=1}^{n} 2x = \frac{1}{n^2(n+1)} n(n+1) = \frac{1}{n}$$

- Finally, $P(X > Y) = p = \frac{n-1}{2n} \approx \frac{1}{2}$ for large n

Definition

- Let (X, Y) have the joint pmf $p(x, y)$ and let $g(X, Y)$ be a function of (X, Y).
- The expectation of $g(X, Y)$ exists if $\sum_x \sum_y |g(x, y)|p(x, y) < \infty$, in which case

$$E]g(X, Y)] = \sum_x \sum_y g(x, y)p(x, y)$$