STAT 516

Bivariate Joint Distributions and Expectations of Functions

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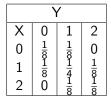
April 12, 2020



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- Consider the experiment of tossing ξ of tossing a fair coin three times.
- Let X be the number of heads among the first two tosses and Y the number of heads among the last two tosses
- ▶ We cannot consider X and Y as two independent Bin(2,0.5) random variables; indeed, if X = 2, then $Y \ge 1$

- The combination of all possible values of (X, Y) are: (0,0), (0,1), (1,0), (0,2), (2,0), (1,1), (1,2), (2,1).
- The joint probability mass function can be described as



- Let (X, Y) be two discrete random variables with sets of values x₁, x₂,... and y₁, y₂,... on a common sample space Ω.
- ▶ The joint pmf of (X, Y) is the function $p(x_i, y_j) = P(X = x_i, Y = y_j)$ for $i, j \ge 1$ and p(x, y) = 0 on any other point (x, y)
- The requirements:
 - 1. $p(x, y) \ge 0 \forall (x, y)$ 2. $\sum_i \sum_j p(x_i, y_j) = 1$

Definition

Let p(x, y) be the joint pmf of (X, Y). The marginal pmf of a function Z = g(X, Y) is

$$p_Z(z) = \sum_{(x,y):g(x,y)=z} p(x,y)$$

In particular,

$$p_X(x) = \sum_y p(x,y)$$

and

$$p_Y(y) = \sum_x p(x, y)$$

Moreover, for any event A

$$P(A) = \sum_{(x,y)\in A} p(x,y)$$

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Let X be the number of aces in the hands of Player 1 and Y the number of aces in the hands of Player 2 in a bridge game

• Then,
$$X, Y \ge 0$$
 and $X + Y \le 4$

- ▶ If Player 1 gets x aces, Player 2 can get y aces in $\binom{4-x}{y}$ ways
- Player 1 also has to get 13 x non-ace cards and Player 2 13 – y non-ace cards

Example I

Thus, the pmf is

$$p(x,y) = \frac{\binom{4}{x}\binom{48}{13-x}\binom{4-x}{y}\binom{35+x}{13-y}}{\binom{52}{13}\binom{39}{13}}$$

where $x \ge 0$, $y \ge 0$, and $x + y \le 4$

For example, p(1,0) = .1249, p(1,1) = .0974, p(1,2) = 0.974, p(1,3) = .0137, and p(1,4) = 0

Summing, we get

$$P(X = 1) = \sum_{y} p(1, y) = .4389$$

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which can also be obtained from the direct formula for the pmf of X :

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$$P(X = 1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} = .4389$$

Let A : X + Y = 2 be the event of interest P(A) = ∑_{(x,y)∈A} p(x,y) = p(0,2) + p(1,1) + p(2,0) = .3901

Note that this probability cannot be computed using only marginal distributions of X and Y

Example II

Let

$$p(x,y)=c(x+y)$$

for $1 \leq x, y \leq n$

▶ To find the constant *c* we need to equate

$$\sum_{x=1}^n \sum_{y=1}^n p(x,y) = 1$$

This results in

$$c\sum_{x=1}^{n}\left[nx+\frac{n(n+1)}{2}\right]=1$$

and, finally,

$$c=\frac{1}{n^2(n+1)}$$

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- Note that this joint pmf is symmetric because x + y = y + x...and, so, p(x, y) = p(y, x)
- That implies, in turn, that X and Y have the same marginal pmf

E.g.

$$p_X(x) = \sum_{y=1}^n p(x, y) = \frac{1}{n^2(n+1)} \sum_{y=1}^n (x+y)$$
$$= \frac{x}{n(n+1)} + \frac{1}{2n}$$

for any $1 \le x \le n$

Example II

- How to compute P(X > Y)
- By symmetry, P(X > Y) = P(Y > X) and so

$$P(X > Y) + P(Y > X) + P(X = Y) = 1$$

• Denote
$$P(X < Y) = P(Y < X) = p$$
; thus,
 $2p + P(X = Y) = 1$ and $p = \frac{1 - P(X = Y)}{2}$

Now,

$$P(X = Y) = \sum_{x=1}^{n} p(x, x) = c \times \sum_{x=1}^{n} 2x = \frac{1}{n^2(n+1)}n(n+1) = \frac{1}{n}$$

Finally, $P(X > Y) = p = \frac{n-1}{2n} \approx \frac{1}{2}$ for large n

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- Let (X, Y) have the joint pmf p(x, y) and let g(X, Y) be a function of (X, Y).
- The expectation of g(X, Y) exists if ∑_x ∑_y |g(x, y)|p(x, y) < ∞, in which case</p>

$$E]g(X,Y)] = \sum_{x} \sum_{y} g(x,y)p(x,y)$$