STAT 516: Continuous random variables: probability density functions, cumulative density function, quantiles, and transformations

Lecture 6: Normal and other unimodal distributions. Functions of continuous random variables

Prof. Michael Levine

February 26, 2020

Symmetry and unimodality

- A density function $f(x)$ is called symmetric around a number M if $f(M+u) = f(M-u)$ for any $u > 0$. A special case is $M = 0$: symmetry around zero when $f(u) = f(-u)$ for any $u > 0$.
- A density function $f(x)$ is called strictly unimodal at M if $f(x)$ is increasing for $x < M$ and decreasing for $x > M$.

Triangular density

- The triangular density where $f(x) = 4x$ for $0 \leq x \leq 0.5$ and $4(1-x)$ for $0.5 \leq x \leq 1$.
- It is symmetric and strictly unimodal

Double Exponential density

- Consider $f(x) = 0.5e^{-x}$ for $x \geq 0$ and $f(x) = 0.5e^{x}$ for $x \geq 0$
- It is symmetric and strictly unimodal with a cusp at $x = 0$
- It can also be written as $f(x) = 0.5e^{|x|}$ for any $-\infty < x < \infty$

Normal distribution

- It is the most important continuous distribution in practice - measurement erros, characteristics of human populations (heights, weights etc.) can be modeled using normal distribution
- Dates all the way back to de Moivre and Laplace
- It is
$$f(x) = f(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
- Typically, we write $X \sim N(\mu, \sigma^2)$; the case $\mu = 0$, $\sigma^2 = 1$ is called the standard normal distribution; its density is
$$\phi(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$
and CDF is $\Phi(x)$

Jacobian formula

- Let $X \sim f(x)$, $f(x)$ is continuous, CDF $F(x)$
- $Y = g(X)$ is a strictly monotone function of X with a non-zero derivative
- $Y \sim f_Y(y)$

$$f_Y(y) = \frac{f(g^{-1}(y))}{|g'(g^{-1}(y))|}$$

Linear Transformation

- Take $g(X) = a + bX$, $b \neq 0$
- Verify that

$$f_Y(y) = \frac{1}{b} f\left(\frac{y-a}{b}\right)$$

  if $b > 0$
- In general, for any $b \neq 0$, it is

$$f_Y(y) = \frac{1}{|b|} f\left(\frac{y-a}{b}\right)$$

Uniform functions are usually not uniform

- $X \sim \mathit{Unif}[0,1]$ and $g(X) = X^2$
- Verify that the pdf of $Y = X^2$ is

$$f_Y(y) = \frac{1}{2\sqrt{y}}$$

for $0 \leq y \leq 1$

Large powers of uniforms

- $X \sim Unif[0,1]$ and $g(X) = X^n$,
- Then,

$$f_Y(y) = \frac{1}{n} y^{\frac{1}{n}-1}$$

  for $0 < y < 1$
- Observe *convergence in probability* of X^n to zero as $n \to \infty$

A function that is not strictly monotone

- Let $X \sim N(0,1)$ and $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ on $(-\infty, \infty)$
- Take $g(X) = X^2$ which is not strictly monotone here
- The direct solution gives

$$f_Y(y) = \frac{e^{-y/2}}{\sqrt{2\pi y}}$$

  for $y > 0$
- This is the density of the **chi-square distribution with one degree of freedom**

General formula for transformations that are not one-to-one

- $X \sim f(x)$, $f(x)$ is continuous, $Y = g(X)$
- For given y, the equation $g(x) = y$ has at most countably many roots, $x_1, x_2, \ldots$. Assume $g^{'}(x_i) \neq 0$
- The pdf of Y is

$$f_Y(y) = \sum_i \frac{f(x_i)}{|g^{'}(x_i)|}$$

From exponential to uniform

- Let $X \sim f(x) = e^{-x}$ for $x > 0$ and zero elsewhere
- Take $Y = g(X) = e^{-X}$
- Verify that

$$f_Y(y) = 1$$

  for any $0 \leq y \leq 1$

General remark: quantile transformation

- For any $X \sim f(x)$ and CDF $F(x)$, the following general result is true
- For X with a continuous cdf $F(x)$, variables $Y = 1 - F(X)$ and $Z = F(X)$ have the $Unif[0,1]$ distribution
- Let U be the $Unif[0,1]$ random variable and $F(x)$ be the continuous CDF
- Then, $X = F^{-1}(U)$ is called the **quantile transformation** of U and it has exactly the CDF F