STAT 516: Examples of Random Variables

Lecture 7: Basic Inequalities of Probability Theory

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Expectation existence and **Markov inequality**

- If $\mathbb{E}X^m$ exists for some positive m $\mathbb{E}X^k$ also exists for any $k \leq m$
- **Markov inequality**: if $u(X)$ is a non-negative function of X s.t. $\mathbb{E}u(X)$ exists, for every positive constant c

$$P[u(X) \geq c] \leq \frac{\mathbb{E}u(X)}{c}$$

Chebyshev inequality

- In Markov inequality, take $u(X) = (X - \mu)^2$ where $\mu = \mathbb{E}X$
- Select $c = k^2\sigma^2$ for a positive k
- The result follows:

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

- In practice, one would select $k \geq 1$ to have a meaningful inequality
- Taking $k\sigma = \varepsilon$ for some $\varepsilon > 0$ obtain a commonly used form

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

How good Chebyshev inequality is in practice?

- Take $X \sim Unif[-\sqrt{3}, \sqrt{3}]$ with $\mu = 0$ and $\sigma^2 = 1$
- For $k = \frac{3}{2}$ the exact probability is

$$P\left(|X| \geq \frac{3}{2}\right) = 1 - \frac{\sqrt{3}}{2}$$

  while the Chebyshev's bound is $\frac{1}{k^2} = \frac{4}{9}$
- Take $k = 2$ and the exact probability is $P(|X| \geq 2) = 0$ while the Chebyshev bound is $\frac{1}{k^2} = \frac{1}{4}$
- Only the existence of the mean and variance of X is assumed - the inequality can be quite conservative!

Example

- Let X be the discrete distribution s.t. $p(-1) = \frac{1}{8}$, $p(0) = \frac{3}{4}$ and $p(1) = \frac{1}{8}$
- Note that $\mu = 0$ and $\sigma^2 = 1$
- If $k = 2$, the exact probability is $P(|X| \geq 1) = \frac{1}{4}$ - the same as the Chebyshev bound
- Cannot improve Chebyshev inequality unless extra assumptions about the distribution of X are made

Chernoff-Bernstein inequality

- A much sharper **large-deviation inequality** is the so-called **Chernoff-Bernstein ineq.**
- Let the random variable X have the mgf $\psi(t) < \infty$ for $t < t_0$ for some $0 < t_0 < \infty$
- Let $\kappa(t) = \log \psi(t)$ be the cumulant generating function
- Define the **rate function** of X $I(x) = \sup_{0<t<t_0}[tx - \kappa(t)]$
- Easy transformation results in

$$P(X \geq x) \leq \exp\left(-I(x)\right)$$

Example

- Take $X \sim N(0, 1)$
- By Chebyshev's inequality,

$$P(X \geq x) = \frac{1}{2} P(|X| \geq x) \leq \frac{1}{2x^2}$$

- By Chernoff-Bernstein inequality, we have

$$P(X \geq x) \leq \exp\left(-x^2/2\right)$$

Convex functions

- A function ϕ on any real interval is **convex** if for any x, y and for any $0 < \gamma < 1$

$$\phi[\gamma x + (1-\gamma)x] \leq \gamma\phi(x) + (1-\gamma)\phi(y)$$

- If the above inequality is strict, the function ϕ is **strictly convex**
- A function ϕ is **concave** if $-\phi$ is convex
- **Strict concavity** is defined in the same way as strict convexity

Jensen's inequality

- For a convex function ϕ on an open interval I and X whose support is contained in I s.t. $\mathbb{E}X < \infty$

$$\phi(\mathbb{E}X) \leq \mathbb{E}[\phi(X)]$$

- If ϕ is strictly convex the inequality is also strict *unless* X is a constant
- The inequality direction is reversed for a concave function

Example

- Let X have a discrete uniform distribution: $P(x) = \frac{1}{n}$ for each of $a_1, \ldots, a_n$ where $a_i > 0$
- $-\log x$ is a convex function so

$$-\log(\mathbb{E}X) \leq \mathbb{E}(-\log X) = -\frac{1}{n}\sum_{i=1}^{n} \log a_i = -\log\left(\prod_{i=1}^{n} a_i\right)^{1/n}$$

- Conclude that

$$(a_1 \ldots a_n)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} a_i$$

Example

- In the previous result, replace a_i by $\frac{1}{a_i}$
- Confirm that

$$\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}} \leq (a_1 a_2 \cdots a_n)^{1/n}$$

- Relationship between the harmonic mean (HM), geometric mean (GM) and the arithmetic mean (AM):

$$HM \leq GM \leq AM$$