STAT 516: Continuous random variables: probability density functions, cumulative density function, quantiles, and transformations

Lecture 6: Continuous random variables

Prof. Michael Levine

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What is a continuous random variable?

- For a real-valued random variable, a function $f(x)$ is the probability density function if, for all $0\infty < a \leq b < \infty$,

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx;$$

;

- In particular, to be a density function, we must have $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
- Note that a pdf may not, in general, be bounded from above since it is not a probability $P(X = x)$!

Cumulative density function of X

- Let X be a continuous random variable with a pdf $f(x)$
- The cdf

$$F(x) = P(X \leq x) = P(X < x) = \int_{-\infty}^{x} f(t)\, dt$$

- At any point x_0 at which $f(x)$ is continuous, the CDF $F(x)$ is differentiable and $F^{'}(x_0) = f(x_0)$.

Independence

- Let $X_1, X_2, \ldots, X_n$ be n random variables defined on the same sample space Ω. We say that $X_1, X_2, \ldots, X_n$ are independent if

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = \prod_{i=1}^{n} P(X_i \leq x_i)$$

  for $\forall x_1, x_2, \ldots, x_n$.
- An equivalent requirement would be

$$P(X_1 \geq x_1, X_2 \geq x_2, \ldots, X_n \geq x_n) = \prod_{i=1}^{n} P(X_i \geq x_i)$$

Density examples

- $f(x) = 1$, if $0 \leq x \leq 1$ and 0 otherwise - a uniform density on $[0, 1]$
- Its CDF is $f(x) = x$ if $0 \leq x \leq 1$, 0 for $x \leq 0$ and 1 for $x \geq 1$
- $f(x) = 3x^2$, if $0 \leq x \leq 1$ and 0 otherwise
- Its CDF is $F(x) = x^3$ if $0 \leq x \leq 1$
- $f(x) = 6x(1 - x)$, if $0 \leq x \leq 1$ and 0 otherwise
- Verify that it is a pdf by direct integration and find its CDF

Using the density to calculate a probability

- Let $X \sim U[0,1]$; consider events $A = \{X \text{ is between } 0.4 \text{ and } 0.6\ \}$ and $B = \{X(1-X) \le .21\}$
- Confirm by direct calculation that $P(A) = .2$
- Check that $x(1-x) = 0.21$ has two roots in $[0,1] : x = 0.3$ and $x = 0.7$. Moreover, $x(1-x) \le .21$ if $x \le .3$ or $x > 0.7$
- Then, $P(B) = P(X \le .3) + P(X \ge 0.7) = 0.6$

Quantiles

- For a continuous random variable $X \sim F(x)$ any $m : F(m) = 0.5$ is a median of X
- Consider $F(x) = 1 - e^{-x}$ for $x \geq 0$ and 0 otherwise and check that it is a CDF
- Find $m = \log 2 = 0.693$