STAT 516: Discrete Random Variables

Lecture 5a: Generating Functions

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Generating Function (GF)

- For a discrete nonnegative integer-valued random variable X define $G(s) = G_X(s) = \mathbb{E}\left(s^X\right) = \sum_{x=0}^{\infty} s^x P(X = x)$
- $G(s)$ is always finite when $s \leq 1$ for any random variable X; it may be finite in other regions as well
- Property 1: Let $G(s)$ be finite in some open interval around the origin. Then, $P(X = k) = \frac{G^{(k)}(0)}{k!}$ for any $k \geq 0$
- Property 2: If $\lim_{s \to 1} G^{(k)}(s)$ is finite, the **kth factorial moment**

$$\mathbb{E}\left[(X(X-1)\dots(X-k+1)\right] = G^{(k)}(1) = \lim_{s \to 1} G^{(k)}(s)$$

GF of a sum of independent RV's; distribution determining property

- Property 1: $X_1, \ldots, X_n$ are independent RV's with GF's $G_1(s) \ldots, G_n(s)$. Then,

$$G_{X_1+\cdots+X_n}(s) = \prod_{i=1}^{n} G_i(s)$$

- The proof is trivial
- Property 2: If two GF's $G(s)$ and $H(s)$ coincide in any nonempty open interval, then X and Y have the same distribution
- The proof is based on a standard property of the power series

Example

- Take X a discrete uniform on $\{1, 2, \ldots, n\}$. Then,

$$G(s) = \mathbb{E}\left[s^X\right] = \frac{1}{n}\sum_{x=1}^{n} s^x = \frac{s(s^n - 1)}{n(s-1)}$$

- Differentiating at 1 and applying the L'Hôpital's rule twice, we find that

$$G^{'}(1) = \frac{n+1}{2}$$

Example

- Take X with a pmf $p(x) = e^{-1}\frac{1}{x!}$, $x = 0, 1, 2, \ldots$.
- Clearly, $p(x) \geq 0$ and $\sum_{x=0}^{\infty} p(x) = 1$
- The generating function is

$$G(s) = \mathbb{E}\left[s^X\right] = \sum_{x=0}^{\infty} s^x e^{-1}\frac{1}{x!} = e^{s-1}$$

- Take the first derivative and conclude that $\mathbb{E}(X) = G^{'}(1) = 1$
- We discovered the **Poisson distribution**...

Moment generating function

- If $M(t) = \mathbb{E}(\exp tX)$ exists in an open neighborhood of size h around zero, it is called a **moment generating function** or **mgf**
- $M(t)$ always exists for $t = 0$; if there exists a radius h s.t. $M(t) < \infty$ for any $t : |t| < h$, many useful properties can be derived
- Also note that the generating function $G(t) = M(\log t)$ whenever $G(t) < \infty$

Main properties

- If the MGF $M(t)$ is finite in some open interval $|t| < h$, then it is infinitely differentiable in that interval and, for any $k \geq 1$,

$$\mathbb{E}X = M^{(k)}(0)$$

- (Distribution determining property)For any two random variables X and Y with existing $M_X(t)$ and $M_Y(t)$ the distributions coincide if and only $M_X(t) = M_Y(t)$ for $t \in (-h, h)$ and $h > 0$
- If $X_1, \ldots, X_n$ are independent random variables, and each X_i has a finite mgf in an open interval around 0, we have

$$M_{X_1+\ldots+X_n}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

Example I

- Let X have the pmf $P(X = x) = \frac{1}{n}$, $x = 1, 2, \dots, n$
- Its mgf is

$$M(t) = \frac{1}{n}\sum_{k=1}^{n} e^{tk} = \frac{e^t(e^{nt} - 1)}{n(e^t - 1)}$$

- To obtain the first derivative at zero; apply L'Hôpital's rule twice to find

$$\mathbb{E}(X) = \frac{n+1}{2}$$

Example II

- Let $X \sim b(1, p)$. The MGF of X is

$$M(t) = pe^t + (1 - p)$$

- Verify that $M^{'}(0) = M^{''}(0) = p$
- Thus, $\mathbb{E}X = \mathbb{E}X^2 = p$

Example III

- For a fair spinner with the numbers 1, 2 and 3 on it let X be the number of spins until the first 3 occurs
- If the spins are independent, we have the geometric distribution
$$p(x) = \frac{1}{3}\left(\frac{2}{3}\right)^{x-1}$$
for $x = 1, 2, 3, \ldots\ldots$
- The mgf of X is
$$M(t) = \mathbb{E}(\exp tX) = \sum_{x=1}^{\infty} \exp(tx)\frac{1}{3}\left(\frac{2}{3}\right)^{x-1} = \frac{1}{3}\left(1 - \exp(t)\frac{2}{3}\right)^{-1}$$
as long as $\frac{2}{3}\exp(t) < 1$ t.i. $t < \log\left(\frac{3}{2}\right)$

Nonexistence of mgf

- Define
$$p(x) = \frac{6}{\pi^2 x^2}$$
for $x = 1, 2, 3, \ldots$ and 0 otherwise
- Easy to see that $\mathbb{E}(tX) = \sum_{x=1}^{\infty} \frac{6 \exp(tx)}{\pi^2 x^2} = +\infty$
- Therefore, this distribution doesn't have an mgf

Nonexistence of mgf II

- A simple cure is to define a **characteristic function** as

$$\phi(t) = \mathbb{E}(itX)$$

- $\phi(t)$ exists for *all* distributions
- $\phi(t)$ also defines the distribution uniquely
- For any existing moment of X we have

$$i^k \mathbb{E} X^k = \phi^{(k)}(0)$$