STAT 516: Basic Probability and its Applications

Lecture 2a: Birthday and Matching Problems

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Birthday Problem

- n unrelated people get together. Each person has an equal probability of being born on any day of the calendar year. Assume that a year consists of 365 days
- What is the probability that we find two or more people with the same birthday?
- There are $(365)^n$ possible birthday choices; we assume that *unrelatedness* rules out any prior knowledge of identical birthdays
- First, find the complement probability of *not* being able to find two or more people with the same birthdays

$$p_n = 1 - \frac{365 \times 364 \times 363 \cdots (366 - n)}{(365)^n} = 1 - \frac{\binom{365}{n} n!}{(365)^n}$$

- For example, $p_{22} = 0.4757$ and $p_{23} = 0.5073$...so, it *only* takes 23 people to be more sure than not there will be people with common birthdays in the group

Stirling's approximation-non-rigorous

- To approximate p_n, using m for 365

$$1 - p_n = \frac{m(m-1)(m-2)\cdots(m-(n-1)}{m^n}$$
$$= 1\left(1 - \frac{1}{m}\right)\cdots\left(1 - \frac{n-1}{m}\right)$$
$$\approx e^{-1/m}e^{-2/m}\cdots e^{-\frac{n-1}{m}} = e^{-\frac{n(n-1)}{2m}} = e^{-\frac{\binom{n}{2}}{m}}$$

Rigorous Stirling's approximation

- Let $\approx$ mean that the ratio converges to 1 as $k \to \infty$. Then,

$$k! \approx e^{-k} k^{k+\frac{1}{2}} \sqrt{2\pi}$$

- Applying this separately to 365! and $(365-n)!$, we have

$$1 - p_n \sim e^{-n} \left(\frac{365}{365-n} \right)^{365-n+\frac{1}{2}}$$

- Since $\log(1+x) \approx x - \frac{x^2}{2}$, the above reduces to

$$\log(1-p_n) \approx -\frac{\binom{n}{2}}{(365-n)}$$

, or,

$$p_n \approx e^{-\frac{\binom{n}{2}}{365}}$$

as above

- For example, this approximation gives $p_{23} \approx 0.5000$

Matching Problem

- n people throw their hats in the air and the wind brings each of them one hat at random. What is the probability that at least one person gets his own hat back?
- $1, 2, \ldots, n$ are arranged in a random manner on a line. If $\pi(i)$ is the number occupying ith location, what is the probability that for at least one i $\pi(i) = i$?
- Let $A_i = \{\pi(i) = i\}, 1 \leq i \leq n$. Need to find $P(\cup_{i=1}^n A_i)$
- Note that $P(A_i) = \frac{1}{n}$, $P(A_i \cap A_j) = \frac{(n-2)!}{n!}, i < j$, $P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!}, i < j < k$ etc

Matching Problem

- By inclusion-exclusion formula,

$$P(\cup_{i=1}^{n} A_i) = \binom{n}{1}\frac{1}{n} - \binom{n}{2}\frac{(n-2)!}{n!} - \cdots + (-1)^{n+1}\frac{1}{n!}$$

$$= 1 - \left[\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

- Using the Taylor series for e^x with $x = -1$, we have

$$e^{-1} \approx \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}$$

  and so

$$p_n = P(\cup_{i=1}^{n} A_i) \approx 1 - e^{-1} = .6321$$

  for large n

- Check that the exact $p_7 = .6321$ so n need not be too large...