# STAT 516: Basic Probability and its Applications

## Lecture 1: Introduction to Probability. Sample spaces, events, probability axioms

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September 3, 2015

# Randome Experiments and a Sample Space

- A **random experiment** can have several possible **outcomes**
- The **sample space** of a random experiment $\mathcal{S}$ is the set of all its possible outcomes.
- Example: tossing the coin once we have the space $\mathcal{S} = \{H, T\}$
- Example: tossing the coin twice and recording the outcomes produces $\mathcal{S} = \{HH, HT, TH, TT\}$
- Example: roll a die with 6 faces and record the outcome. Then, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.

# The idea of probability

- For us, **probability** is the idealized quantity the **relative frequency** approaches as the number of experiments grows
- Other approaches are possible...e.g. treating the probability as as personal (subjective) statement of beliefs
- The theory does not change

# Basic Set theory I

- A **set** is a collection of objects
- Example: if $\mathbb{R}$ is a set of all real numbers , then 0.5 is an **element** of $\mathbb{R}$
- Notation: $0.5 \in \mathbb{R}$
- If $C = \{x : 0 \leq x \leq 1\}$, we say that $0.5 \in C$ as well
- A set $C$ is **countable** if it can be **enumerated**

# Examples

- Neither $C$ nor $\mathbb{R}$ are countable
- The set $\{1, 2, 3, 4, 5\}$ is countable
- The set of all integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is also countable

# Basic Set theory II

- $A$ is a subset of $B$ if for every $x \in A$ we have $x \in B$
- Notation: $A \subset B$
- If $A \subset B$ and $B \subset A$, then $A = B$
- A set $A$ is an **empty set** if it has no elements.
- Notation: $A = \varnothing$

# Some Relations from Set Theory

- The **union** of two events $A$ and $B$ is the event consisting of all outcomes that are *either* in $A$ or in $B$ or in *both* events. Notation: $A \cup B$. Reading: *A or B*
- The **intersection** of two events $A$ and $B$ is the event consisting of all outcomes that are in *both A and B*. Notation $A \cap B$. It is read "*A and B*".
- The **complement** of an event $A$, denoted by $A'$, is the set of all outcomes in $\mathcal{S}$ that are *not* in $A$.

# Some Relations from the Set Theory

- The set $A \triangle B$ consists of all outcomes that are either in $A$ or in $B$ but *not in both*. Reading: a **symmetric difference** of $A$ and $B$
- The **difference** between two sets $A$ and $B$ or a **relative complement of** $B$ **in** $A$ $A \setminus B = A \cap B'$ consists of all outcomes in $A$ that are *not* in $B$
- The notation $A - B$ is also used but is less preferred because it can also mean the set of all $\{a - b | a \in A, b \in B\}$
- As an example, $A \cup B \cup C$ means at least one of $A$, $B$ or $C$ occurred
- $A \cap B \cap C$ means that all of $A$, $B$ and $C$ occurred
- $A \cap B' \cap C'$ means that $A$ occurred but not $B$ or $C$

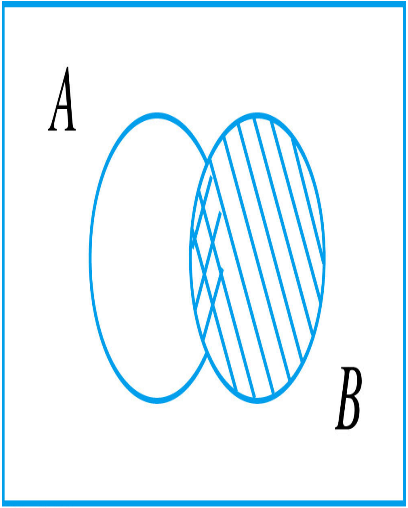
# Examples

- If $C_1 = \varnothing$, $C_1 \cup C_2 = C_2$ for any $C_2$
- For any $C$ $C \cup C = C$
- $C_1 = \{1, 2, 3\}$ and $C_2 = \{2, 3, 4\}$
- $C_1 \cup C_2 = \{1, 2, 3, 4\}$ and $C_1 \cap C_2 = \{2, 3\}$
- Consider rolling a die with the sample space $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.
- $A^{'} = \{4, 5, 6\}$.

# Some Additional Set Theory Relations

- For a sample space $\mathcal{S}$, an element $x \in \cup_k A_k$ if and only if there exists $k_0$ such that $x \in A_{k_0}$.
- $x \in \cap_k A_k$ if and only $x \in A_k$ for all $k$
- Example: if $A_k = \{0, 1, 2, ..., k\}$ then $\cup_k A_k = \{0, 1, 2, 3, ...\}$ and $\cap_k A_k = \{0, 1\}$
- Example: let $C_k = \left\{x : \frac{1}{k+1} \leq x \leq 1\right\}$, $k = 1, 2, 3, \ldots$. Then, $\cup_{k=1}^{\infty} C_k = \{x : 0 \leq x \leq 1\}$
- Example: let $C_k = \left\{x : 0 < x < \frac{1}{k}\right\}$ for $k = 1, 2, 3 \ldots$. Then, $\cap_{k=1}^{\infty} C_k = \varnothing$.

# Venn Diagrams



# Some operation laws on sets

- **Distributive laws**:

$$B \cap (\cup_k A_k) = \cup_k (B \cap A_k)$$

and

$$B \cup (\cap_k A_k) = \cap_k (B \cup A_k)$$

- **De Morgan's laws**: for any two sets $A_1$ and $A_2$

$$[A_1 \cap A_2]' = A_1' \cup A_2'$$

and

$$[A_1 \cup A_2]' = A_1' \cap A_2'$$

# Set functions

- **Set functions** are important in probability!
- Example: $C$ is a set in a two-dimensional space and $Q(C)$ is its area if $C$ is finite
- For a circle $C = \{(x, y) : x^2 + y^2 \leq 1\}$ $Q(C) = \pi$
- For a set $C = \{(0, 0), (1, 1)\}$ $Q(C) = 0$
- For a rectangular triangle $C = \{(x, y) : 0 \leq x, 0 \leq y, x + y \leq 1\}$ $Q(C) = \frac{1}{2}$

# Summation for set functions

- Let
$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^x & \text{if } x = 1, 2, 3, \ldots \\ 0 & \text{elsewhere} \end{cases}$$
- For $C = \{0 \leq x \leq 3\}$,
$$Q(C) = \sum_C f(x) = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 = \frac{7}{8}$$

# Integration for set functions

- Let $Q(C) = \int_C \exp(-x)\, dx$
- For $C = \{x : 0 \leq x < \infty\}$

$$Q(C) = \int_0^\infty \exp(-x)\, dx = 1$$