STAT 516: Some discrete distributions

Lecture 12: Poisson distribution

Prof. Michael Levine

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Definition

- $X \sim P(m)$ if
$$p(x) = \frac{m^x e^{-m}}{x!}$$
for $x = 0, 1, 2 \ldots$
- Easy to verify that $p(x)$ is a pmf
- Unlike the other discrete distributions cannot be described using a simple algorithm

Poisson assumptions

- Let $g(x, w)$ be the probability of x changes in an interval of length w
- The following are Poisson postulates:
  1. $g(1, h) = \lambda h + o(h)$ for a positive λ
  2. $\sum_{x=2}^{\infty} g(x, h) = o(h)$
  3. Numbers of changes in non-overlapping intervals are independent
- A process satisfying these assumptions is known as a **Poisson process**

Interpretation

1. The probability of one change is proportional to the length of an interval
2. The probability of more than two changes is very small
3. Self-explanatory

Heuristic derivation I

- The probability of no events in an interval of length $w + h$ is

$$g(0, w+h) = g(0, w)[1 - \lambda h - o(h)]$$

- Rearrange to obtain

$$\frac{g(0, w+h) - g(0, w)}{h} = -\lambda g(0, w) - \frac{o(h) g(0, w)}{h}$$

- Letting $h \to 0$ obtain $g^{'}(0, w) = -\lambda g(0, w)$
- Since $g(0, 0) = 1$ the solution is

$$g(0, w) = e^{-\lambda w}$$

Heuristic derivation II

- For $x > 0$,

$$g(x, w+h) = [g(x, w)][1-\lambda h-o(h)]+[g(x-1, w)][\lambda h+o(h)]+o(h)$$

- Rearranging terms, we have

$$\frac{g(x, w+h) - g(x, w)}{h} = -\lambda g(x, w) + \lambda g(x, 1-w) + \frac{o(h)}{h}$$

- As $h \to 0$, we have $g_w^{'}(x, w) = -\lambda g(x, w) + \lambda g(x-1, w)$
- By mathematical induction with the initial condition $g(x, 0) = 0$,

$$g(x, w) = \frac{(\lambda w)^x e^{-\lambda w}}{x!}$$

for $x = 0, 1, 2, \ldots$
- The number of changes in an interval of length w is $X \sim P(\lambda w)$

Moment generating function, mean and variance of the Poisson

- Easily computed:
$$M(t) = e^{m(e^t-1)}$$
- Verify that for $X \sim P(m)$ $\mathbb{E}\, X = Var\, X = m$ - the mean is equal to the variance
- Due to the above, a common notation is $X \sim P(\mu)$

Example

- April receives on average three phone calls per day at her home. What is the percentage of days on which she receives no phone calls?
- Clearly, if Poisson model is true, $P(X = 0) = e^{-3} = 0.0498$
- What is the percentage of days she receives more than five phone calls?
- Again, under the Poisson assumption, $P(X > 5) = 1 - P(X \leq 5) = 1 - \sum_{x=0}^{5} e^{-3}3^x/x! = 0.0839$
- If using R, use dpois(0,3) in the first case and 1-ppois(5,3) in the second
- Keep in mind that the Poisson model is not the only one possible!

Example

- Lengths of an electronic tape contain, on average, 1 defect per 100 ft. What is the probability that the 50 ft long tape will be defect free?
- Assuming the homogeneous Poisson process, we model it as $X \sim P(0.5)$
- Thus, $P(X = 0) = e^{-.5} = .6065$

Additive property

- For independent $X_1, \ldots, X_n \sim P(m_i)$
  $Y = \sum_{i=1}^n X_i \sim P(\sum_{i=1}^n m_i)$
- Can prove in one line using the moment generating function of X but can also be done directly
- Recall other distributions with the additive property - binomial and negative binomial

Example

- The probability of a blemish in 1 foot of wire is about 0.001 while the probability of two or more blemishes is infinitesimally small
- If X is the number of blemishes in 3000 feet of wire, model it as $X \sim P(3000 * 0.001) = P(3)$
- Now select three bails of wire at random...what is the probability that the mean number of blemishes is at least 5?
- Clearly, $\bar{Y} = \sum_{i=1}^{3} X_i \sim P(9)$
- Thus,

$$P(\bar{X} \geq 5) = P(Y \geq 15) = 1 - P(Y \leq 14) = 0.041$$