STAT 516: Some Special Continuous Distributions

Lecture 7: Uniform, Exponential, Gamma, Inverse Gamma, and Beta Distributions

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Uniform Distribution

- Let X have the pdf $f(x) = \frac{1}{b-a}$ for any $a \leq x \leq b$ and 0 elsewhere. We say that $X \sim U[a,b]$ - uniformly distributed on $[a,b]$
- Main properties:
  1. If $X \sim U[0,1]$, then $a + (b-a)X \sim U[a,b]$; if $X \sim U[a,b]$, then $\frac{X-a}{b-a} \sim U[0,1]$
  2. The CDF of $U[a,b]$ is $F(x) = \frac{x-a}{b-a}$ for any $a \leq x \leq b$, equal to zero if $x < 0$ and equal to 1 if $x > 1$.
  3. The mgf of $U[a,b]$ is $\psi(t) = \frac{e^{tb}-e^{ta}}{(b-a)t}$
  4. The mean and the variance of the $U[a,b]$ are

$$\mu = \frac{a+b}{2};\ \sigma^2 = \frac{(b-a)^2}{12}$$

Example

- The diameters (measured in centimeters) of circular strips made by a machine are uniform in the interval $[0, 2]$. Strips with an area larger than 3.1 cm^2 cannot be used. Suppose that 200 strips are made in one shift.
- The are of a circular strip of a radius r is πr^2. If $r \sim U[0, 1]$ we have

$$p = P(\pi r^2 > 3.1) = P(r^2 > 3.1/\pi) = P(r^2 > .9868)$$
$$= P(r > .9934) = .0066$$

- Thus, the number of strips among 200 that cannot be used have the Bin$(200, 0.0066)$ distribution
- Their expected number is $200 * 0.0066 = 1.32$

Exponential Distribution

- A nonnegative random variable X has the exponential distribution with parameter $\lambda > 0$ if its pdf $f(x) = \frac{1}{\lambda}e^{-x/\lambda}$ for $x > 0$.
- We write $X \sim Exp(\lambda)$.
- The case $\lambda = 1$ is called the standard exponential density

Exponential Distribution: Main Properties

- For $X \sim Exp(\lambda)$
  1. $\frac{X}{\lambda} \sim Exp(1)$
  2. The CDF is $F(x) = 1 - e^{-x/\lambda}$ for $x > 0$
  3. $E(X^n) = \lambda^n n!$ for $n \geq 1$
  4. The mgf is $\psi(t) = \frac{1}{1-\lambda t}$ defined for $t < \frac{1}{\lambda}$

Example I

- For $X \sim Exp(4)$ find $P(X > 4)$
- Since $\frac{X}{4} \sim Exp(1)$

$$P(X > 4) = \int_1^{\infty} e^{-x}\, dx = e^{-1} = 0.3679$$

- Thus $EX \neq \tilde{\mu}(X) = 4 \log 2$

Example II

- Let $X \sim Exp(\lambda)$ and $s, t > 0$. Then, one can show directly that
$$P(X > s + t | X > s) = P(X > t)$$
- Thus, the exponential distribution is the continuous analog of the geometric distribution

Gamma distribution

- A generalization of the exponential distribution with a mode usually at some $m > 0$ is the Gamma distribution
- $X \sim G(\alpha, \lambda)$ if its pdf

$$f(x|\alpha, \lambda) = \frac{e^{-x/\lambda} x^{\alpha-1}}{\lambda^{\alpha}\Gamma(\alpha)}$$

  for any $x > 0$ and $\alpha, \lambda > 0$
- When $\alpha = 1$ we have the exponential distribution as a special case

Main Properties

1. The CDF of $X \sim G(\alpha, \lambda)$ is
$$F(x) = \frac{\gamma(\alpha, x/\lambda)}{\Gamma(\alpha)}$$
where $\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1}\, dt$ is the incomplete Gamma function
2. The nth moment is
$$E(X^n) = \lambda^n \frac{\Gamma(\alpha + n)}{\gamma(\alpha)}$$
for any $n \geq 1$
3. The MGF is
$$\psi(t) = (1 - \lambda t)^{-\alpha}$$
for any $t < \frac{1}{\lambda}$
4. The mean and the variance are
$$E(X) = \alpha\lambda; \;\; \sigma^2 = \alpha\lambda^2$$

Corollary

- Let $X_1, \ldots, X_n$ be independent $Exp(\lambda)$ variables. Then, $X_1 + \cdots + X_n \sim G(n, \lambda)$
- The proof follows by calculating the mgf of the sum:

$$E(e^{t(X_1+\cdots+X_n)}) = E(e^{tX_1}e^{tX_2}\cdots e^{tX_n}) = (1-\lambda t)^{-n}$$

Example I

- 40 people have been invited to a party. The amount of diet soda each guest receives is distributed as $Exp(8)$ (in ounces).
- Two bottles of soda with of 200 oz each are available
- What is the probability that the supply will fall short of the demand?

Example I

- Let $n = 40$ and X_i the amount consumed by *ith* guest.
- Then, $X_1 + \cdots + X_n \sim G(n, 8)$ and so the needed probability is

$$P(X_1 + \cdots + X_n > 400) = 1 - P(X_1 + \cdots + X_n \leq 400)$$
$$= 1 - \frac{\gamma(40, 400/8)}{\Gamma(40)} = 0.065$$

Example II

- If $X_1, \ldots, X_m$ are m independent $N(0,1)$, then $T = \sum_{i=1}^{m} X_i^2 \sim G\left(\frac{m}{2}, 2\right)$ with the pdf

$$f_m(t) = \frac{e^{-t/2} t^{m/2-1}}{2^{m/2}\Gamma\left(\frac{m}{2}\right)}$$

  for any $t > 0$
- This is the χ^2-density with m degrees of freedom: $T \sim \chi^2_m$.
- Clearly, the mean of this distribution is m and the variance is $2m$

Sample variance for iid normal observations

- If $X_1, \ldots, X_n$ are independent with the same variance σ^2, we have the sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

- S^2 is the unbiased estimator of σ^2 :

$$E(S^2) = \sigma^2$$

- Indeed, just note that

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$$

- Then,

$$E(S^2) = \frac{1}{n-1}[n(\sigma^2 + \mu^2) - n(\sigma^2/n + \mu^2)] = \sigma^2$$

χ^2 distribution and the sample variance

- Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. Then,

$$\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$$

The inverse Gamma distribution

- Heavily used as a prior in Bayesian statistics and in reliability theory
- If $X \sim G(\alpha, \lambda)$ then $\frac{1}{X}$ has the inverse Gamma distribution
- $g(X) = \frac{1}{X}$ is a strictly monotone function for positive X
- Its inverse function is $g^{-1}(y) = \frac{1}{y}$ and $g^{'}(x) = -\frac{1}{x^2}$
- Thus, Y has the density

$$f_Y(y) = \frac{f\left(\frac{1}{y}\right)}{|g^{'}\left(\frac{1}{y}\right)|}$$

$$= \frac{e^{-1/(\lambda y)} y^{-1-\alpha}}{\lambda^{\alpha}\Gamma(\alpha)}$$

for any positive y

- This density is heavily skewed for small α
- The right tail is very heavy: there is no finite mean when $\alpha < 1$

Example: simulating a Gamma variable

- Note that the CDF of a Gamma random variable does not have a closed form expression
- Nevertheless, when the first parameter is an integer, say $\alpha = n$, it is easy to generate values from it
- Consider $Exp(1)$ random variable X. Its CDF is $F(x) = 1 - e^{-x}$.
- Thus, its inverse is the quantile function

$$F^{-1}(p) = -\log(1-p)$$

  for any $0 < p < 1$
- Therefore, for any $U \sim U[0,1]$ we have $-\log(1-U) \sim Exp(1)$ and

$$Y = -\sum_{i=1}^{n} \log(1-U_i) = -\log \prod_{i=1}^{n} (1-U_i) \sim G(n,1)$$

- Finally, if the second parameter needed is $\lambda \neq 1$, the answer is $Y' = \lambda Y$

Beta distribution

- Beta distribution arose as a generalization of the uniform distribution
- It is also defined on a closed interval but can be either decreasing or increasing, symmetric and unimodal or unimodal and assymetric etc.
- In other words, it is capable of taking a wide variety of shapes

Definition and basic properties

- We say that $X \sim Be(\alpha, \beta)$ if its pdf is

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$$

where $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx$ and the parameters $\alpha > 0, \beta > 0$.

Definition and basic properties

- The beta function $B(\alpha, \beta)$ can also be represented as

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

- The CDF is

$$F(x) = \frac{B_x(\alpha, \beta)}{B(\alpha, \beta)}$$

  and $B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\, dt$ is an incomplete beta function
- The nth moment is

$$E(X^n) = \frac{\Gamma(\alpha + n)\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + n)\Gamma(\alpha)}$$

- The mean and variance are

$$\mu = \frac{\alpha}{\alpha + \beta} \; \sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Remarks

- Note that the specific case $\alpha = 1$, $\beta = 1$ results in a uniform density on $[0, 1]$
- Proof of the CDF formula is just a restatement of the definition of the incomplete beta function
- To prove the nth moment formula, one has simply to note that

$$B(\alpha + n, \beta) = \int_0^1 x^{\alpha+n-1}(1-x)^{\beta-1}\, dx$$

Example

- Suppose a standardized one hour exam takes 45 min on average to finish and the standard deviation of the finishing times is 10 min. What percentage of examinees finish in less than 40 min?
- If we assume Beta distribution for the finishing time, we can use formulas for the mean and the variance
- Thus,

$$\frac{\alpha}{\alpha+\beta} = \frac{3}{4}$$

  and

$$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{36}$$

- The result is $\alpha = 4.32$ and $\beta = 1.44$
- Finally, the answer is given by

$$P\left(X < \frac{2}{3}\right) = \frac{\int_0^{2/3} x^{3.32}(1-x)^{.44}\,dx}{B(4.32, 1.44)} = 0.281$$

Mixture of two beta densities

- It is, unfortunately, impossible for a beta distribution to be bimodal in $[0, 1]$ for any values of α and β
- To circumvent this, we can use a suitable **mixture** of two Betan densities
- Consider
$$f(x) = 0.5f_1(x) + 0.5f_2(x)$$
where $f_1(x)$ and $f_2(x)$ are densities of $Be(6, 2)$ and $Be(2, 6)$, respectively
- Note that $f(x)$ is clearly a density;
$$f(x) = 21x(1-x)[x^4 + (1-x)^4]$$
for $0 \leq x \leq 1$
- Verify that this density is bimodal