1. Suppose that in a certain population 30% of couples have one child, 50% have two children, and 20% have three children. One family is picked at random from this population. What is the expected number of boys in the family? Assume that the probability of a childbirth resulting in a boy is 0.5.

   Solution: let Y be the number of children in the family that was picked, and let X be the number of boys it has. Making the usual assumption of a childbirth being equally likely to be a boy or a girl,

   $$E(X) = E_Y[E(X|Y = y)] = .3 \times .5 + .5 \times 1 + .2 \times 1.5 = .95$$

2. Let the random variables X and Y have the joint pmf

   (a) $p(x, y) = \frac{1}{3}$ for $(x, y) = (0, 0), (1, 1), (2, 2)$ and zero elsewhere

   (b) $p(x, y) = \frac{1}{3}$ for $(x, y) = (0, 2), (1, 1), (2, 0)$ and zero elsewhere

   Find the correlation $\rho_{X,Y}$ in both cases. Solution: as an example, consider (2a). To solve this question, we simply use the definition of the covariance as $Cov(X, Y) = E(XY) - E(X)E(Y)$. Recall that, in order to find e.g. $E(X)$ you should find the marginal pmf of X first. Since $p_X(x) = \sum_y p(x, y)$ we have $p_X(0) = p_X(1) = p_X(3) = \frac{1}{3}$ and so $E(X) = 1$. Due to the symmetry of the joint distribution, the marginal distribution of Y is the same and so $E(Y) = E(X) = 1$. Using the joint distribution, one finds $E(XY) = \frac{1}{3} + 4 \times \frac{1}{3} = \frac{5}{3}$. Thus, $Cov(X, Y) = \frac{5}{3} - 1 = \frac{2}{3}$. Next, $Var(X) = E(X^2) - (EX)^2 = \frac{5}{3} - 1 = \frac{2}{3}$ and the same is true for $Var(Y)$ due to symmetry. By definition, $\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = 1$. Following exactly the same steps, one finds that in (2b) $\rho_{X,Y} = -1$.

3. Let U and V be two random variables with common mean and common variance. Let $X = U + V$ and $Y = U - V$. Show that X and Y are uncorrelated.

   Solution: by definition, $Cov(X, Y) = E(U + V)(U - V) - E(U + V)E(U - V) = EU^2 - EV^2 - E(U + V)E(U - V) = 0$

4. Consider the trinomial distribution with pmf

   $$P(X = x, Y = y) = \frac{n!}{x!y!(n - x - y)!} p_1^x p_2^y p_3^{n-x-y}$$

   where x, y are non-negative integers such that $x + y \leq n$. Of course, $p_1, p_2, p_3 \geq 0$ and $p_1 + p_2 + p_3 = 1$.

(a) Find the conditional distribution $P(Y = y|X = x)$

(b) What is $E(Y|X = x)$? Can you find $E(X|Y = y)$ without any additional computations?

Solution: the marginal $P(X = x) = \binom{n}{x} p_1^x (1 - p_1)^{n-x}$ f or $x = 0, 1, 2, \ldots, n$. Therefore,

$$P(Y = y|X = x) = \frac{(n - x)!}{y!(n - x - y)!} \left(\frac{p_2}{1 - p_1}\right)^y \left(\frac{p_3}{1 - p_1}\right)^{n-x-y}$$

for any $y = 0, 1, 2, \ldots, n - x$ and 0 otherwise. In other words, $Y|X = x \sim Bin(n - x, p_2/(1 - p_1))$. Thus, $E(Y|X = x) = (n - x)\frac{p_2}{1-p_1}$. By symmetry, we have $E(X|Y = y) = (n - y)\frac{p_1}{1-p_2}$.