1. A sprinter covers on average 140 cm, with a standard deviation of 5 cm, in each stride. What is the approximate probability that this runner will cover the 100 m distance in 70 or fewer steps?

   Solution: denote the distance covered by the sprinter in n strides by $X_1, X_2, \ldots X_n$ and assume that they are independent. Each X_i has the mean $\mu = 140$ and $\sigma^2 = 25$. By the CLT, $S_n = \sum_{i=1}^{n} X_i$ is approx $N(140n, 25n)$. Since 100m is the same as $10,000$ cm in n strides, we have

$$P(S_{70} \geq 10,000) = 1 - P(S_{70} < 10,000)$$
$$\approx 1 - \Phi\left(\frac{10,000 - 140 \times 70}{\sqrt{25 * 70}}\right) = 1 - \Phi(4.78) \approx 0$$

2. Suppose a fair die is rolled twice. Let X and Y be the larger and the smaller of the two rolls (note that X can be equal to Y). Each of X and Y takes the individual values of $1, \ldots, 6$ and, of course, $X \geq Y$. As an example, by direct counting, $p(X = 2, Y = 1) = p(2, 1) = \frac{2}{36}$.

   (a) Write down the entire joint pmf of (X, Y) in the form of a table

   (b) What is $P(X = 1)$ and $P(X = 2)$?

   Solution:

   (a) The pmf is

	Y					
X	1	2	3	4	5	6
1	$\frac{1}{36}$	0	0	0	0	0
2	$\frac{1}{18}$	$\frac{1}{36}$	0	0	0	0
3	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	0	0	0
4	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	0	0
5	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	0
6	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$

   (b) $P(X = 1) = \sum_{y=1}^{6} P(X = 1, Y = y) = \frac{1}{36}$ and $P(X = 2) = \sum_{y=1}^{6} P(X = 2, Y = y) = \frac{1}{12}$

3. In the experiment of three tosses of a fair coin, we have worked out (in the lecture slides) the joint pmf of (X, Y) where X is the number of heads in the first two tosses and Y is the number of heads in the last two tosses. Find the entire conditional distribution of X given $Y = 0$.

Solution: $P(X=0|Y=0) = \frac{p(0,0)}{p_Y(0)} = \frac{1/8}{1/4} = \frac{1}{2}$; $P(X=1|Y=0) = \frac{p(1,0)}{p_Y(0)} = \frac{1/8}{1/4} = \frac{1}{2}$; $P(X=2|Y=0) = \frac{p(2,0)}{p_Y(0)} = \frac{0}{1/4} = 0$.

4. Consider again the example of the joint distribution of the maximum and the minimum of two rolls of a fair die. Let X be the maximum and Y be the minimum. Find the value of $E(X|Y=y)$ for all y. How does the conditional expectation behave as a function of y?

   Solution: e.g.

$$E(X|Y=1) = \frac{1 \times 1/36 + 1/18 * [2 + \cdots + 6]}{\frac{1}{36} + \frac{5}{18}} = \frac{41}{11} = 3.73$$

   or

$$E(X|Y=3) = \frac{3 \times 1/36 + (1/18) * 15}{\frac{1}{36} + \frac{3}{18}} = \frac{33}{7} = 4.71$$

   and

$$E(X|Y=5) = \frac{5 \times 1/36 + (1/18) * 6}{\frac{1}{36} + \frac{1}{18}} = \frac{17}{3} = 5.77$$

   The conditional expectation is an increasing function of y and this seems to make sense.