Homework 3

1. A random variable X has the moment generating function $\psi(t) = \frac{1}{1-t}$ defined for any $t < 1$. What is the probability $P(X < 1)$?

   Solution: by inspection of its mgf, $X \sim Exp(1)$ with the pdf $f(x) = e^{-x}$ for any $x > 0$ and 0 elsewhere. Thus, $P(X < 1) = \int_0^1 e^{-x}\,dx = 1 - e^{-1}$.

2. Suppose X and Y are independent random variables and $X \sim G(\alpha, \lambda)$, $Y \sim G(\beta, \lambda)$. Find the distribution of $X+Y$ using moment generating functions.

   Solution: the moment generating function of $X+Y$ is $\psi(t) = \frac{1}{(1-\lambda t)^\alpha} \times \frac{1}{(1-\lambda t)^\beta} = \frac{1}{(1-\lambda t)^{\alpha+\beta}}$. Note that both moment generating functions of X and Y are defined for any $t < \frac{1}{\lambda}$ and so is the resulting mgf function $\psi(t)$. Thus, we conclude that $X + Y \sim G(\alpha + \beta, \lambda)$.

3. Suppose $X \sim G(\alpha, \lambda)$. Find the expected value $E\left(\frac{1}{X}\right)$.

   Solution: the simplest approach is simply to think in terms of the function $g(x) = \frac{1}{x}$ and compute the integral

$$E\left(\frac{1}{X}\right) = \int_0^\infty \frac{1}{x} \frac{x^{\alpha-1}e^{-x/\lambda}}{\lambda^\alpha \Gamma(\alpha)}\,dx$$
$$= \frac{1}{\lambda(\alpha-1)} \times \frac{1}{\lambda^{\alpha-1}\Gamma(\alpha-1)} \int_0^\infty x^{(\alpha-1)-1} e^{-x/\lambda}\,dx = \frac{1}{\lambda(\alpha-1)}$$

   since the remaining expression is simply the integral of the $\Gamma(\alpha-1, \lambda)$ density and is thus equal to 1. The above calculations only work if $\alpha > 1$ since the integral does not converge for other values of α. For any $\alpha \leq 1$, we can only say that $E\left(\frac{1}{X}\right) = \infty$. An alternative way is to view $\frac{1}{X}$ as an inverse Gaussian random variable and use the pdf of this density that we derived in our slides. The answer, of course, should be the same.

4. Let $Z \sim N(0, 1)$. Find $P\left(0.5 < \left|Z - \frac{1}{2}\right| < 1.5\right)$.

   Solution: first, if $Z \geq \frac{1}{2}$, we have $\left|Z - \frac{1}{2}\right| = Z - \frac{1}{2}$ and so the corresponding probability is $P(1 < Z < 2) = F(2) - F(1) = 0.13591$. The second case is when $Z < \frac{1}{2}$ which implies that $\left|Z - \frac{1}{2}\right| = \frac{1}{2} - Z$ and the corresponding probability is $P(-1 < Z < 0) = F(0) - F(-1) = 0.5 - 0.15866 = 0.34134$. Thus,

$$P\left(0.5 < \left|Z - \frac{1}{2}\right| < 1.5\right) = 0.13591 + 0.34134 = 0.47725$$