STAT 516 MIDTERM; 40 pt total

1. (10pt) What is the probability that in a game of bridge a randomly selected hand of 13 cards contains 2 spades, 7 hearts, 3 diamonds and 1 club?

   Solution:

$$\frac{\binom{13}{2}\binom{13}{7}\binom{13}{3}\binom{13}{1}}{\binom{52}{13}}$$

2. (10pt)

   (a) (5pt)Consider families with two children, and assume that all four possible distributions of sex - BB, BG, GB, GG - are equally likely. Let E be the event that a family has at most one girl and F, the event that a family has children of both sexes. Are the events E and F independent? Although many people buy

   (b) (5pt) Now consider families with three children; assume that each of the eight possible sex distributions is equally likely. Are the events E and F independent?

   Solution:

   (a) Clearly, $P(E) = \frac{3}{4}$ and $P(F) = \frac{1}{2}$. At the same time, $P(EF) = \frac{1}{2}$ and E and F are not independent

   (b) Now, $P(E) = \frac{4}{8}$ and $P(F) = \frac{6}{8}$; $P(EF) = \frac{3}{8} = P(E)P(F)$ and E and F are independent

3. (10pt) Let X be a geometric random variable with the probability of success p where $0 < p < 1$. We define its pmf as

$$p(x) = P(X = x) = p(1-p)^x$$

where $k = 0, 1, 2, \ldots$.

(a) (5pt)Compute the probability generating function $G(s)$ of this distribution and specify its radius of convergencve

(b) Using $G(s)$ you just obtained, compute the expectation and the variance of the random variable X

Solution:

(a) By direct computation,

$$G(s) = \sum_{x=0}^{\infty} s^x p(1-p)^x = p\frac{1}{1 - s(1-p)}$$

when $|s| \leq 1$

(b) $P'(s) = \frac{p(1-p)}{(1-s(1-p)^2}$ and $P''(s) = \frac{2p(1-p)^2}{(1-s(1-p))^3}$. Thus, $EX = \frac{1-p}{p}$, $VarX = \frac{1-p}{p^2}$

4. (10pt) A fair die is rolled n times. What is the number of trials needed for the probability of at least one 6 to be greater than or equal to $\frac{1}{2}$?

   Solution; the solution is given by the smallest integer n such that

$$1 - \left(\frac{5}{6}\right)^n \geq \frac{1}{2}$$

   In other words, $n \geq \frac{\log 2}{\log 1.2} \approx 3.8$