Homework 3

1. Consider the experiment of rolling a fair die twice, and let X be the sum of the two rolls. Then X takes the values $x_1 = 2$, $x_2 = 3$,...$x_{11} = 12$.

   - Find the probabilities of all the possible values of X by direct counting and put them in the form of a table
   - What is the median of the resulting distribution?

   Solution: the resulting table is

x	2	3	4	5	6	7	8	9	10	11	12
p(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

   The median of this distribution is 7.

2. Suppose X has the pmf $p(x) = \frac{c}{1+x^2}$, $x = 0, \pm1, \pm2, \pm3$. Find the distribution of the random variable $Z = \sin\left(\frac{\pi}{2}X\right)$.

   Solution: direct evaluation of the constant c results in $c = \frac{5}{13}$. The possible values of Z are

x	z
-3	1
-2	0
-1	-1
0	0
1	1
2	0
3	-1

   Thus, $P(Z = 0) = P(X = -2) + P(X = 0) + P(X = 2) = \frac{7}{5}c = \frac{7}{13}$. The full pmf of $Z = h(X)$

z	-1	0	1
P(Z=z)	3/13	7/13	3/13

3. Consider the random variable X - the number of aces in one of the player's hand in a game of bridge. Note that X can take any of the values $x = 0, 1, 2, 3, 4$ while the remaining $13 - x$ cards in this player's hand must be non-ace cards.

(a) What is the pmf of X?

(b) Compute the expected value of X

Solution: the pdf is

$$p(x) = \frac{\binom{4}{x}\binom{48}{13-x}}{\binom{52}{13}}$$

for $x = 0, 1, 2, 3, 4$. Direct calculation of the expected value gives $E(X) = 1$. This seems sensible - there are four aces in the deck and we should expect that, on average, one ace will go to each of the four players.

4. Suppose a fair coin is tossed n times. Let X be the number of heads obtained; then, $n - X$ is the number of tails obtained. Define $W = g(X) = \max\{X, n - X\}$. Compute the expectation EW for the case of $n = 4$.

   Solution: by direct counting, it is easy to find the pmf

x	0	1	2	3	4
p(x)	1/16	4/16	6/16	4/16	1/16

   Then, $EW = E[\max\{X, 4 - x\}] = \sum_{x=0}^{4} g(x)p(x) = 2.75$. Note that this number is not equal to the expectation of either X or Y.