1. Let (X, Y) have the joint density function $f(x, y) = c - 2(c-1)(x + y - 2xy)$ for any $x, y \in [0, 1]$. It is possible that this function is non-negative for any $0 < c < 2$.

   (a) Show that $f(x, y)$ integrates up to 1 for any c and is, therefore, a density function Solution: by direct integration,

   $$\int_0^1 \int_0^1 f(x,y)\,dxdy = c - 2(c-1)\int_0^1 (x + y - 2xy)\,dxdy$$
   $$= c - 2(c-1)\int_0^1 \left(\frac{1}{2} + y - y\right)\,dy = c - (c-1) = 1$$

   (b) Find marginal densities of X and Y. Are X and Y are independent? Solution: the marginal density of X

   $$f_1(x) = \int_0^1 f(x,y)dy = c - 2(c-1)\left[x + \frac{1}{2} - x\right] = 1$$

   for any $x \in [0, 1]$. Since $f(x, y)$ is symmetric in x and y, the marginal density of Y $f_2(y)$ is also uniform on $[0, 1]$. Clearly, their product is not equal to the original joint pdf $f(x, y)$.

2. Suppose (X, Y) has the joint density $f(x, y) = 6xy^2$ for any $x, y \geq 0$ and $x + y \leq 1$. Thus, it is yet another density on the triangle with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$. Find the probability $P\left(X + Y < \frac{1}{2}\right)$.

   Solution: by definition,

   $$P\left(X + Y < \frac{1}{2}\right) = \int_{(x,y);x,y\geq 0, x+y<\frac{1}{2}} 6xy^2\,dxdy$$
   $$= 6\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-y} xy^2\,dxdy$$
   $$= 6\int_0^{\frac{1}{2}} y^2 \frac{\left(\frac{1}{2}-y\right)^2}{2}\,dy$$
   $$= 3\int_0^{\frac{1}{2}} y^2\left(\frac{1}{2}-y\right)^2\,dy = \frac{1}{320}$$

3. Suppose a point (x, y) is picked at random from the unit circle. We want to find its expected distance from the center of the circle. In other words, we have (X, Y) with the joint density $f(x, y) = \frac{1}{\pi}$ for any $x^2 + y^2 \leq 1$ and zero otherwise. Hint: use polar coordinates.

Solution: we need to find $E(\sqrt{X^2+Y^2})$. By definition,

$$E(\sqrt{X^2+Y^2}) = \frac{1}{\pi}\int_{(x,y):x^2+y^2\leq 1} \sqrt{x^2+y^2}\,dxdy$$

Transforming to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and $dxdy = r\,drd\theta$, we have

$$E(\sqrt{X^2+Y^2}) = \frac{1}{\pi}\int_{(x,y):x^2+y^2\leq 1} \sqrt{x^2+y^2}\,dxdy$$
$$= \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi} r^2\,d\theta dr = 2\int_0^1 r^2\,dr = \frac{2}{3}$$