STAT 511

Lecture 12: Confidence Intervals
Devore: Section 7.1-7.2

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October 22, 2018

Motivation

- Why do we need a confidence interval? Because with each new sample we have a new parameter estimate (e.g. new sample mean)....
- Which one do we choose? We do not know the true mean μ and do not know how close each one is to μ.
- Thus, we want to have some degree of precision reported together with an estimate
- Suppose our $\bar{X} = 10$. We want to say something like..."With probability 95% the true mean is between 9 and 11"

Basic properties of Confidence Intervals

- Consider normal population distribution with known σ
- We want to estimate unknown μ
- The problem is purely illustrative; in practice, mean is usually known before the variance (standard deviation)
- We know that $\bar{X}$ is normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$.

- Because the area under the normal curve between -1.96 and 1.96 is 0.95, we have

$$P(-1.96 \leq Z \leq 1.96) = P\left(-1.96 \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq 1.96\right) = 0.95$$

- Simple algebra tells us that

$$P\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

The meaning of the confidence interval

- The event in parentheses above is a random interval with the left endpoint $\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}$ and right endpoint $\bar{X} + 1.96\frac{\sigma}{\sqrt{n}}$. It is centered at sample mean $\bar{X}$.
- For a given sample $X_1 = x_1, \ldots, X_n = x_n$, we compute the observed sample mean $\bar{x}$ and substitute it in the definition of our random interval instead of $\bar{X}$. The resulting fixed interval is called 95% *confidence interval* (CI).
- The usual way to express it is either to say that

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

is a 95% CI for μ

- Alternatively, we say that

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}$$

  with 95%
- A more concise expression is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$

Example

- The average zinc concentration recovered from a sample of zinc measurements in 36 different locations is found to be 2.6 grams per milliliter. Find the 95% confidence interval for the mean zinc concentration in the river, assuming it is normally distributed. The population standard deviation is known to be 0.3
- μ is estimated by $\bar{x} = 2.6$. The z-value we need is 1.96. Hence, the 95% confidence interval is

$$2.6 - 1.96\frac{0.3}{\sqrt{36}} < \mu < 2.6 + 1.96\frac{0.3}{\sqrt{36}}$$

- The above reduces to $2.50 < \mu < 2.70$.

Other Levels of Confidence

- Any desired level of confidence can be achieved by changing the critical z-value used in constructing the interval.
- A $100(1-\alpha)\%$ confidence interval for the mean μ of a normal population for the known value σ is

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

or, equivalently, by $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$

Precision and Sample Size Choice

- It is easy to understand that the confidence interval width is $2z_{\alpha/2} \cdot \sigma/\sqrt{n}$. Clearly, the more precise confidence interval we require, the wider it has to be.
- In other words, the confidence level (reliability) is inversely related to precision. The usual strategy is to specify both the confidence level and the interval width and then determine the necessary sample size.
- Consider the response time that is normally distributed with standard deviation 25 millisec. What sample size is necessary to ensure that 95% CI has a width of (at most) 10?

- Clearly, the sample size must satisfy $10 = 2 \cdot (1.96)(25/\sqrt{n})$.
- Therefore, $n = \frac{4 \cdot (1.96)^2 \cdot 525}{100} = 96.04$
- In practice, we would require $n = 97$.
- Thus, to ensure an interval width w we need to have

$$n = \left(2z_{\alpha/2} \cdot \frac{\sigma}{w}\right)^2$$

- The half-width of a 95% interval $1.96\sigma/\sqrt{n}$ is sometimes called the *bound on the error of estimation*.

Large-Sample Confidence Intervals

- If X has the mean μ and variance σ^2, for large enough n,

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

has approximately standard normal distribution, according to CLT

- But we do not know σ! What do we do?
- Solution: use $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$. It can be verified that

$$Z = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has approximately standard normal distribution for large n.

- This implies that
$$\bar{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$
is a **large-sample confidence interval** with confidence level approximately $100(1-\alpha)\%$
- This result is valid regardless of the true distribution of X

Example

- The alternating current breakdown voltage of an insulator indicates its dielectric strength. We have $n = 48$ observations of breakdown voltage of a particular circuit under certain conditions. The mean is $\bar{x} = 54.7$ and $s = 5.23$. Then, the 95% confidence interval is

$$54.7 \pm 1.96\sqrt{5.23}/\sqrt{48} = 54.7 \pm 1.5 = (53.2, 56.2)$$

- The final result is $53.2 < \mu < 56.2$

Some Remarks

- The choice of n to be considered "large enough" is different according to different textbooks. The most common is $n > 40$.
- In the case of large-sample confidence interval the choice of sample size is more difficult. The reason is that you do not know s before you actually sample your data.
- The best solution is to try to guess s and to err on the side of caution by choosing larger s.

A Confidence Interval for a Population Proportion

- Assume X is the number of "successes" in the sample of size n. Denote p the proportion of successes in the overall population.
- If n is small compared to the population size, X is binomial with mean np and variance $np(1-p)$.
- The natural estimator of p is $\hat{p} = X/n$, the sample fraction of success
- $\hat{p}$ has an approximately normal distribution; its mean is p and the standard deviation is $\sqrt{p(1-p)/n}$.
- Standardizing, we have

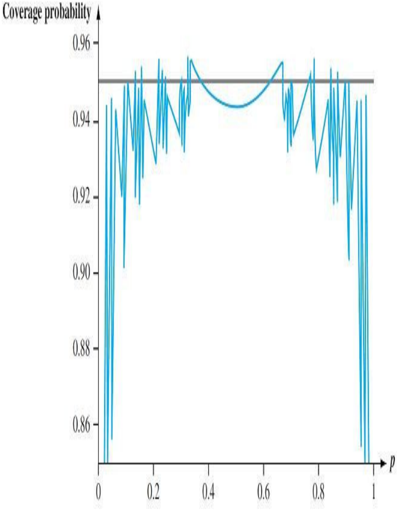
$$P\left(-z_{\alpha/2} < \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} < z_{\alpha/2}\right) \approx 1-\alpha$$

- As before, the confidence interval can be easily derived replacing $<$ by $=$ and solving the quadratic equation for p.
- The resulting confidence interval is (a, b) where

- b is

$$b = \frac{\hat{p} + z^2_{\alpha/2}/2n + z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n + z^2_{\alpha/2}/4n^2}}{1 + (z^2_{\alpha/2})/n}$$

- For large n, approximation $\hat{p} \pm z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}$ was traditionally used.
- However, its use is not recommended now due to problems concerning true coverage


Coverage probability
0.96
0.94
0.92
0.90
0.88
0.86
p
0
0.2
0.4
0.6
0.8
1

Example

- In $n = 48$ trials in a laboratory, 16 resulted in ignition of a particular type of substrate by a lighted cigarette. Let p denote the probability of "success" (long-run proportion). A point estimate for p is $\hat{p} = \frac{16}{48} = .333$. A confidence interval for p with a confidence level of about 95% is

$$\frac{.333 + (1.96)^2/96 \pm 1.96\sqrt{(.333)(.667)/48 + (1.96)^2/9216}}{1 + (1.96)^2/48}$$
$$= (.217, .474)$$

One-sided confidence intervals

- As an example, a reliability engineer may want only a lower confidence bound for the true average lifetime of a certain component.
- To derive a $100(1-\alpha)\%$ one-sided CI, we use

$$P\left(\frac{\bar{X}-\mu}{S/\sqrt{n}} < z_\alpha\right) \approx 1-\alpha$$

- Thus, the large-sample upper confidence bound is

$$\mu < \bar{x} + z_\alpha \cdot \frac{s}{\sqrt{n}}$$

while the large-sample lower confidence bound for μ is

$$\mu > \bar{x} - z_\alpha \cdot \frac{s}{\sqrt{n}}$$

Example I

- In order to claim that a gas additive increases mileage, an advertiser must fund an independent study in which n vehicles are tested to see how far they can drive, first without and then with the additive. Let X_i denote the increase in mpg observed for vehicle i and let $\mu = E\,X_i$.
- A large corporation makes an additive with $\mu = 1.01$ mpg. The respective study involves $n = 900$ vehicles in which $\bar{x} = 1.01$ and $s = 0.1$ are observed
- If the significance level is $\alpha = 0.05$, $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} < 1.645$ and the one-sided CI is

$$\mu_0 > 1.01 - 1.645\frac{0.1}{\sqrt{900}} = 1.0045$$

Example II

- An amateur automotive mechanic invents an additive that increases mileage by an average of $\mu = 1.21$ mpg. The mechanic funds a small study of $n = 9$ vehicles in which $\bar{x} = 1.21$ and $s = 0.4$ are observed.
- The one-sided CI is

$$\mu_0 > 1.21 - 1.645 \frac{0.4}{\sqrt{9}} = 0.9967$$