STAT 511

Lecture 4: Counting Techniques, Conditional Probability and Independence

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Why are counting techniques needed?

- Sample space is too large
- The outcomes are not equally likely

The product rule

- Ordered pairs: if the first element can be selected in n_1 ways and the second in n_2 ways we have $n_1 n_2$ possible ordered pairs.
- Note that this is sampling **with replacement**.
- An ordered **k-tuple** is an ordered collection of k objects.
- Example: $\{1, 2, 3, 4, 5, 6\}$ is an ordered 6-tuple.
- If each of i elements can be selected in n_i ways, the total number of ordered k-tuples possible is $n_1 n_2 \times \ldots \times n_k$

Examples

- Each of the 2 clinics has 3 specialists in internal medicine, 2 general surgeons. You require the services of both and benefit most by choosing both from the same clinic. How many opportunities do you have to do it?
- Answer: $4 \times 3 = 12$
- What if you also have 2 pediatricians for each clinic and need to choose a pediatrician as well?
- Answer: $4 \times 3 \times 2 = 24$.

Example

- Consider a system of five identical components connected in a series. Let F be a failure and S be a success for any single component. What is the probability of an event $A = \{\text{system fails}\}$?
- Answer: $A' = \{S, S, S, S, S\}$, thus, $P(A) = 1 - P(A') = 1 - P(S)^5$.
- E.g. if $P(S) = 0.9$, $P(A) = 1 - 0.9^5 = 0.41$.

Sampling without replacement

- An ordered sequence of k objects taken from a set of n distinct objects is called a **permutation** of size k. The notation is $P_{k,n}$.
- The number of possible permutations is

$$P_{k,n} = n(n-1)(n-2) \times \ldots \times (n-(k-1))$$

- For any positive integer m,

$$m! = m(m-1) \times \ldots \times 1.$$

- Using factorial notation, we have

$$P_{k,n} = \frac{n!}{(n-k)!}.$$

Example

- A boy has 4 beads red, white, blue, and yellow. In how many different ways can three of the beads be strung together in a row?
- This is a permutation since the beads will be in a row (order). Thus, the answer is

$$P_{3,4} = \frac{4!}{(4-3)!} = 4! = 24.$$

Combinations

- Given a set of n distinct objects, any *unordered* subset of size k of the objects is called a **combination.**
- The usual notations is either $C_{k,n}$ or $\binom{n}{k}$
- To compute $\binom{n}{k}$, note that it is

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}.$$

Example

- Three balls are selected at random without replacement from the jar containing 8 balls- 2 red balls, 3 blacks and 3 greens. Find the probability that one ball is red and two are black
- The answer

$$\frac{\binom{2}{1}\binom{3}{2}}{\binom{8}{3}} = \frac{3}{28}.$$

Conditional Probability

- For any two events A and B with $P(B) > 0$, the **conditional probability** of A given that B has occurred is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- A direct consequence is the **multiplication rule**:

$$P(A \cap B) = P(A|B) \cdot P(B)$$

Interpretation of conditional probability

- Conditional probability results from restricting the outcomes to only those that are *inside* B
- Therefore, instead of

$$P(A) = \frac{N(A)}{N(S)}$$

we have

$$\frac{N(A \cap B)}{N(S \cap B)} = \frac{P(A \cap B)}{P(S \cap B)} = \frac{P(A \cap B)}{P(B)}$$

which is exactly our definition of conditional probability

Example

- Consider the situation where of all individuals buying a digital camera 60% include an optional memory card, 40% - an extra battery and 30% - both. What is the probability that a person who buys an extra battery also buys a memory card?
- Let $A = \{\text{memory card purchased }\}$ and $B = \{\text{battery purchased }\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.4} = 0.75$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.6} = 0.50$$

Example

- Four individuals have responded to a request by a blood bank for blood donations. Their blood types are unknown. Suppose only type O+ is desired and only one of the four actually has this type
- If the potential donors are selected in random order for typing, what is the probability that at least three individuals must be typed to obtain the desired type?
- Let $B =$ first type not O+ and $A =$ second type not O+ ; clearly, $P(B) = \frac{3}{4}$
- Given that the first type is not O+, two of the three individuals left are not O+, so $P(A|B) = \frac{2}{3}$
- By multiplication rule, the probability that at least three individuals are typed is

$$P(A \cap B) = P(A|B)P(B) = 0.5$$

Bayes' Theorem and its origins: an example

- HIV testing: four outcomes are $D \cap +$, $D \cap -$, $D^{'} \cap +$ and $D^{'} \cap -$
- The **prevalence** of the disease $P(D)$ is commonly known; say, $P(D) = 0.001$
- $+|D^{'}$ is a **false positive** and $-|D$ is a **false negative**.
- Diagnostic procedures undergo extensive evaluation and, therefore, probabilities of false positives and false negatives are commonly known
- We assume that $P(+|D^{'}) = 0.015$ and $P(-|D) = 0.003$; for more details, see E.M. Sloan et al. (1991) "HIV testing: state of the art", JAMA, 266:2861-2866

Bayes' Theorem and its origins: an example

- The quantity of most interest is usually the **predictive value** of the test $P(D|+)$
- By definition of conditional probability and multiplication rule, we have

$$P(D|+) = \frac{P(D \cap +)}{P(+)} = \frac{P(D \cap +)}{P(D \cap +) + P(D' \cap +)}$$
$$= \frac{P(D) * P(+|D)}{P(D) * P(+|D) + P(D) * P(-|D)}$$
$$= \frac{P(D) * [1 - P(-|D)]}{P(D) * [1 - P(-|D)] + P(D) * P(-|D)}$$

- The resulting probability will be small because false positives are much more common then false negatives in the general population

- Law of the total probability:

$$P(B) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$$

for mutually exclusive $A_1, \ldots, A_k$ such that

$$\cup_{i=1}^{k} A_i = S$$

- Now we can state Bayes' theorem
- If $P(A_i) > 0$ for any $i = 1, \ldots, k$, then for any B such that $P(B) > 0$

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)}$$

Example

- A store stocks light bulbs from three suppliers. Suppliers A, B, and C supply 10%, 20%, and 70% of the bulbs respectively.
- It has been determined that company A's bulbs are 1% defective while company B's are 3% defective and company C's are 4% defective.
- If a bulb is selected at random and found to be defective, what is the probability that it came from supplier B?

- Let $D =$ defective lightbulb

$$
\begin{aligned}
P(B|D) &= \frac{P(B)P(D|B)}{P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)} \\
&= \frac{0.2(0.03)}{0.1(0.01) + 0.2(0.03) + 0.7(0.04)} \approx 0.1714
\end{aligned}
$$

Independence

- Two event A and B are independent events iff(which means "if and only if")
$$P(A|B) = P(A)$$
- An obvious alternative definition is that A and B are independent iff
$$P(A \cap B) = P(A) \cdot P(B)$$
- If A and B are independent, so are B and A
- The independence of A and B implies that
  1. A and B' are independent
  2. A' and B are independent
  3. A' and B' are independent

Example

- Consider the fair six-sided die. Define $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$ and $C = \{1, 2, 3, 4\}$.
- Clearly, $P(A) = \frac{1}{2}$, $P(A|B) = \frac{1}{3}$ and $P(A|C) = \frac{1}{2}$.
- Therefore, A and C are independent but A and B are NOT!

Some consequences of the definition of independence

- If A and B are mutually exclusive events, they **cannot** be independent.
- If either $P(A) = 0$ or $P(B) = 0$, A and B are always independent:

$$0 \leq P(A \cap B) \leq min(P(A), P(B)) = 0,$$

therefore $P(A \cap B) = 0$

Practical considerations

- In practice, we do not verify independence of events. Instead, we ask ourselves whether independence is a property that we wish to incorporate into a mathematical model of an experiment, based on the common sense
- Thus, independence is commonly *assumed*
- Example: let A=(A student is a female) and B=(A student is concentrating in elementary education); clearly, $P(B|A) \neq P(B)$

Mutual independence

- $A_1, \ldots, A_n$ are mutually independent if for every $k = 2, 3, \ldots,$ and every possible subset of indices $i_1, i_2, \ldots, i_k$

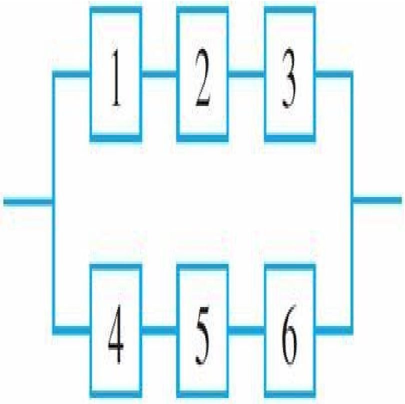
$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k})$$

- Mutual independence is a much stronger condition than pairwise independence!
- Example I: let a single outcome be ω_1 and the entire sample space is $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$
- Then let $A = \{\omega_1, \omega_2\}$, $B = \{\omega_1, \omega_3\}$ and $C = \{\omega_1, \omega_4\}$
- If each outcome is equally likely with $p = \frac{1}{4}$, it is easy to check that pairwise independence is valid
- However,

$$P(A \cap B \cap C) = \frac{1}{4} \neq \left(\frac{1}{2}\right)^3 = P(A)P(B)P(C)$$

Example

- The article “Reliability Evaluation of Solar Photovoltaic Arrays”(Solar Energy, 2002: 129141) presents various configurations of solar photovoltaic arrays consisting of crystalline silicon solar cells



Example

- Consider a particular lifetime value t_0; we want to determine the probability that the system lifetime exceeds t_0
- Let A_i denote the event that the lifetime of cell i exceeds t_0 $(i = 1, 2, ..., 6)$.
- A_i's are independent events and $P(A_i) = 0.9$

Example

- Then, the probability of the event of interest is

$$
\begin{aligned}
&P([A_1 \cap A_2 \cap A_3] \cup A_4 \cap A_5 \cap A_6]) \\
&= P(A_1 \cap A_2 \cap A_3) + P(A_4 \cap A_5 \cap A_6) \\
&- P([A_1 \cap A_2 \cap A_3] \cap A_4 \cap A_5 \cap A_6]) \\
&= (0.9)^3 + (0.9)^3 - (0.9)^6 = .927
\end{aligned}
$$

- Consider sample space Ω of 36 ordered pairs (i,j) with $i,j = 1,\ldots,6$
- We assume that for each pair the probability of occurring $p = \frac{1}{36}$
- Let $A = \{(i,j) : j = 1,2 \text{ or } 5\}$, $B = \{(i,j) : j = 4,5 \text{ or } 6\}$ and $C = \{(i,j) : i+j = 9\}$
- Then,

$$P(A \cap B) = \frac{1}{6} \neq \frac{1}{4} = P(A)P(B)$$

$$P(A \cap C) = \frac{1}{36} \neq \frac{1}{18} = P(A)P(C)$$

and

$$P(B \cap C) = \frac{1}{12} \neq \frac{1}{18} = P(B)P(C)$$

- However,

$$P(A \cap B \cap C) = \frac{1}{36} = P(A)P(B)P(C)$$