STAT 511

Lecture 3: Introduction to Probability. Sample spaces, events, probability axioms

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Sample Space

- The *sample space* of an experiment $\mathcal{S}$ is the set of all its possible outcomes.
- Example: tossing the coin once we have the space $\mathcal{S} = \{H, T\}$
- Example: tossing the coin twice and recording the outcomes produces $\mathcal{S} = \{HH, HT, TH, TT\}$
- Example: roll a die with 6 faces and record the outcome. Then, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.

Simple and Compound Events

- An *event* is a collection (subset) of outcomes contained in the sample subspace. It is *simple* if it consists of exactly one outcome and *compound* otherwise.
- Example: consider recording the freeway exit (L or R) for each of the 3 vehicles at the end of the exit ramp. The eight possible outcomes form the sample space $\mathcal{S} = \{LLL, RLL, LRL, LLR, LRR, RLR, RRL, RRR\}$.
- Examples of simple events would be:
  - $E_1 = LLL$
  - $E_2 = LRL$

Examples of Compound Events

- Examples of compound events would be:
  - Exactly one vehicle turns left $A = \{LRR, RLR, RRL\}$
  - All three vehicles turn in the same direction $B = \{LLL, RRR\}$

Some Relations from Set Theory

- The union of two events A and B is the event consisting of all outcomes that are *either* in A or in B or in *both* events. Notation: $A \cup B$. Reading: *A or B*
- The intersection of two events A and B is the event consisting of all outcomes that are in *both A and B*. Notation $A \cap B$. It is read "*A and B*".
- The *complement* of an event A, denoted by A', is the set of all outcomes in $\mathcal{S}$ that are *not* in A.

Example

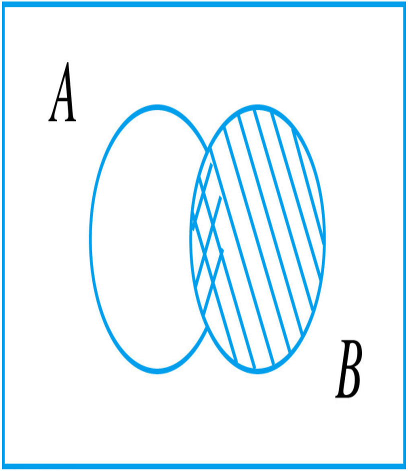
- Consider rolling a die with the sample space $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.
- Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$. Then
  - $A \cup B = \{1, 2, 3, 5\}$.
  - $A \cap B = \{1, 3\}$.
  - $A' = \{4, 5, 6\}$.

Disjoint Events

- When A and B have no outcomes in common, they are said to be **mutually exclusive** or **disjoint** events.
- We can also say that then intersection of A and B is the **null event**: $A \cap B = \varnothing$. Here, $\varnothing$ stands for the event with no outcomes in it.
- Example. When rolling a die, the event $A = \{2, 4, 6\}$ (evens) and $B = \{1, 3, 5\}$ (odds) are mutually exclusive

Disjoint events

- When pulling a single card from a standard deck of cards, events A =heart, diamond(red) and B =spade, club(black) are mutually exclusive.



Some Additional Set Theory Relations

- For a sample space S, an element $x \in \cup_k A_k$ if and only if there exists k_0 such that $x \in A_{k_0}$.
- $x \in \cap_k A_k$ if and only $x \in A_k$ for all k
- As an example, if $A_k = \{0, 1, 2, ..., k\}$ then $\cup_k A_k = \{0, 1, 2, 3, ...\}$ and $\cap_k A_k = \{0\}$
- Distributive laws:

  $$B \cap (\cup_k A_k) = \cup_k (B \cap A_k)$$

  and

  $$B \cup (\cap_k A_k) = \cap_k (B \cup A_k)$$

- De Morgan's laws: for any two sets A_1 and A_2

  $$[A_1 \cap A_2]' = A_1' \cup A_2'$$

  and

  $$[A_1 \cup A_2]' = A_1' \cap A_2'$$

Axioms of Probability

1. Axiom 1:
$$P(A) \geq 0$$
for any event A.
2. Axiom 2:
$$P(\mathcal{S}) = 1.$$
3. Axiom 3: for disjoint $A_1, \ldots, A_n$
$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{k=1}^{n} P(A_k)$$
and
$$P(A_1 \cup A_2 \cup \ldots) = \sum_{k=1}^{\infty} P(A_k).$$

Properties of Probability

- $P(A^{'}) = 1 - P(A)$
- If A and B are mutually exclusive, then $P(A \cap B) = 0$
- For any A and B $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example

- In a residential suburb, 60% of all households get Internet service from a local cable company, 80% get television service from that company, and 50% get both services from that company
- The probability that a randomly selected household subscribes to at least one of these two services from the local company is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.9$$

Example

- The probability that a household subscribes only to TV service is

$$P(A^{'} \cap B) = P(A \cup B) - P(A) = 0.9 - 0.6 = 0.3,$$

that it subscribes only to Internet is

$$P(A \cap B^{'}) = P(A \cup B) - P(B) = 0.1,$$

and the probability that a household subscribes to exactly one of these services from the local company is
$P(A^{'} \cap B) + P(A \cap B^{'}) = 0.1 + 0.3 = 0.4$

Systematic Determination of Probabilities and Equally Likely Outcomes

- If a sample space is either finite or countable, it is sensible to assign probabilities to each simple (single outcome) event E_i first
- For any compound event A we have

$$P(A) = \sum_{\text{all } E_i's \text{ in } A} P(E_i)$$

Systematic Determination of Probabilities and Equally Likely Outcomes

- Now let an experiment have N possible outcomes. If all of the outcomes are equally likely with probability p, $1 = \sum_{i=1}^{N} P(E_i) = p * N$ and so

$$p = \frac{1}{N}$$

- For a particular compound event A that consists of $N(A)$ outcomes,

$$P(A) = \frac{N(A)}{N}$$

Example

- During off-peak hours a commuter train has five cars. Suppose a commuter is twice as likely to select the middle car #3 as to select either adjacent car (#2 or #4) , and is twice as likely to select either adjacent car as to select either end car (#1 or #5).
- If the probability of selecting car i is p_i, we have $p_3 = 2p_2 = 2p_4$, and $p_2 = 2p_1 = 2p_5 = p_4$
- Therefore, $1 = \sum p_i = 10p_1$ and $p_1 = p_5 = 0.1$, $p_2 = p_4 = 0.2$ and $p_3 = 0.4$.
- The probability that one of the three middle cars is selected is, then, $p_2 + p_3 + p_4 = 0.8$.

Example

- Consider a card that is drawn from a well-shuffled deck of 52 playing cards. What is the probability that it is a queen or a heart?
  1. $P(Q) = \frac{4}{52}$ and $P(H) = \frac{13}{52}$
  2. $P(Q \cap H) = \frac{1}{52}$
  3. $P(Q \cup H) = P(Q) + P(H) - P(Q \cap H) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$