HW6 solution

7.2.1 Recall that for the model discussed in Example 7.1.1, the posterior distribution of θ was Beta$(n\bar{x}+\alpha, n(1-\bar{x})+\beta)$. The posterior density is then given by

$$\pi_{\theta|x_1,..,x_n} = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(n\bar{x}+\alpha)\Gamma(n(1-\bar{x})+\beta)}\theta^{n\bar{x}+\alpha-1}(1-\theta)^{n(1-\bar{x})+\beta-1}$$

The posterior mean is given by

$$\begin{aligned} &E(\theta^m \mid x_1, ..., x_n) \\ &= \int_0^1 \frac{\Gamma(\alpha+\beta+n)}{\Gamma(n\bar{x}+\alpha)\Gamma(n(1-\bar{x})+\beta)}\theta^{n\bar{x}+\alpha+m-1}(1-\theta)^{n(1-\bar{x})+\beta-1}\,d\theta \\ &= \frac{\Gamma(\alpha+\beta+n)\Gamma(n\bar{x}+\alpha+m)}{\Gamma(n\bar{x}+\alpha)\Gamma(\alpha+\beta+n+m)}. \end{aligned}$$

7.2.2 Recall that for the model discussed in Example 7.1.2 the posterior distribution of μ is

$$N\left(\left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\left(\frac{\mu_0}{\tau_0^2}+\frac{n}{\sigma_0^2}\bar{x}\right), \left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\right)$$

By exercise 2.6.3, the posterior distribution of the third quartile $\Psi = \mu + \sigma_0 z_{0.75}$ is

$$N\left(\left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\left(\frac{\mu_0}{\tau_0^2}+\frac{n}{\sigma_0^2}\bar{x}\right)+\sigma_0 z_{0.75}, \left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\right)$$

Since the normal distribution is symmetric about its mode and the mean exists, the posterior mode and mean agree and given by

$$\hat{\psi} = \left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\left(\frac{\mu_0}{\tau_0^2}+\frac{n}{\sigma_0^2}\bar{x}\right)+\sigma_0 z_{0.75}.$$

7.2.6 Recall that the posterior distribution of θ in Example 7.2.2 is Beta$(n\bar{x}+\alpha, n(1-\bar{x})+\beta)$. To find the posterior variance we need only to find the second moment as follows.

$$
\begin{aligned}
&E\left(\theta^2 \mid x_1, ..., x_n\right) \\
&= \int_0^1 \theta^2 \frac{\Gamma(n+\alpha+\beta)}{\Gamma(n\bar{x}+\alpha)\Gamma(n(1-\bar{x})+\beta)} \theta^{n\bar{x}+\alpha-1}(1-\theta)^{n(1-\bar{x})+\beta-1}\, d\theta \\
&= \frac{\Gamma(n+\alpha+\beta)}{\Gamma(n\bar{x}+\alpha)\Gamma(n(1-\bar{x})+\beta)} \int_0^1 \theta^{n\bar{x}+\alpha+1}(1-\theta)^{n(1-\bar{x})+\beta-1}\, d\theta \\
&= \frac{\Gamma(n+\alpha+\beta)}{\Gamma(n\bar{x}+\alpha)\Gamma(n(1-\bar{x})+\beta)} \frac{\Gamma(n\bar{x}+\alpha+2)\Gamma(n(1-\bar{x})+\beta)}{\Gamma(n+\alpha+\beta+2)} \\
&= \frac{(n\bar{x}+\alpha+1)(n\bar{x}+\alpha)}{(n+\alpha+\beta+1)(n+\alpha+\beta)}
\end{aligned}
$$

The posterior variance is then given by

$$
\begin{aligned}
\operatorname{Var}(\theta \mid x_1, ..., x_n) &= E\left(\theta^2 \mid x_1, ..., x_n\right) - \left(E(\theta \mid x_1, ..., x_n)\right)^2 \\
&= \frac{(n\bar{x}+\alpha+1)(n\bar{x}+\alpha)}{(n+\alpha+\beta+1)(n+\alpha+\beta)} - \left(\frac{n\bar{x}+\alpha}{n+\alpha+\beta}\right)^2 \\
&= \frac{(n\bar{x}+\alpha)(n(1-\bar{x})+\beta)}{(n+\alpha+\beta+1)(n+\alpha+\beta)^2}.
\end{aligned}
$$

Now $0 \le \bar{x} \le 1$, so

$$
\begin{aligned}
\operatorname{Var}(\theta \mid x_1, ..., x_n) &= \frac{(n\bar{x}+\alpha)(n(1-\bar{x})+\beta)}{(n+\alpha+\beta+1)(n+\alpha+\beta)^2} \\
&\le \frac{(1+\alpha/n)(1+\beta/n)}{n(1+\alpha/n+\beta/n+1/n)(1+\alpha/n+\beta/n)^2} \to 0
\end{aligned}
$$

as $n \to \infty$.

7.2.10 The likelihood function is given by $L(\lambda \,|\, x_1, ...x_n) = \lambda^n e^{-n\bar{x}\lambda}$. The prior distribution has density given by $\beta_0^{\alpha_0} \lambda^{\alpha_0 - 1} e^{-\beta_0 \lambda} / \Gamma(\alpha_0)$. The posterior density of λ is then given by $\pi(\lambda \,|\, x_1, ...x_n) \propto \lambda^{n+\alpha_0-1} e^{-\lambda(n\bar{x}+\beta_0)}$, and we recognize this as being the density of a Gamma$(n + \alpha_0, n\bar{x} + \beta_0)$ distribution. The posterior mean and variance of λ are then given by $E(\lambda \,|\, x_1, ...x_n) = (n + \alpha_0) / (n\bar{x} + \beta_0)$, $Var(\lambda \,|\, x_1, ...x_n) = (n + \alpha_0) / (n\bar{x} + \beta_0)^2$.

To find the posterior mode we need to maximize $\ln\left(\lambda^{n+\alpha_0-1} e^{-\lambda(n\bar{x}+\beta_0)}\right) = (\alpha_0 + n - 1) \ln \lambda - \lambda (n\bar{x} + \beta_0)$. This has first derivative given by $(\alpha_0 + n - 1) / \lambda - (n\bar{x} + \beta_0)$ and second derivative $-(\alpha_0 + n - 1) / \lambda^2$. Setting the first derivative equal to 0 and solving gives the solution $\hat{\lambda} = (\alpha_0 + n - 1) / (n\bar{x} + \beta_0)$. The second derivative at this value is $-(n\bar{x} + \beta_0)^2 / (\alpha_0 + n - 1)$, which is clearly negative, so $\hat{\lambda}$ is the unique posterior mode.