HW3 solution

6.2.1 The MLEs are $\hat{\theta}(1) = a, \hat{\theta}(2) = b,\ \hat{\theta}(3) = b, \hat{\theta}(4) = a$.

6.2.2 The likelihood function is given by $L(\theta \mid x_1, ..., x_n) = \theta^{n\bar{x}}(1-\theta)^{n(1-\bar{x})}$. The log-likelihood function is given by $l(\theta \mid x_1, ..., x_n) = n\bar{x}\ln\theta + n(1-\bar{x})\ln(1-\theta)$. The score function is given by

$$S(\theta \mid x_1, ..., x_n) = \frac{n\bar{x}}{\theta} - \frac{n(1-\bar{x})}{1-\theta}$$

Solving the score equation gives $\hat{\theta}(x_1, ..., x_n) = \bar{x}$. Note that since $0 \le \bar{x} \le 1$ we have that

$$\left.\frac{\partial S(\theta \mid x_1, ..., x_n)}{\partial \theta}\right|_{\theta=\bar{x}} = \left.-\frac{n\bar{x}}{\theta^2} - \frac{n(1-\bar{x})}{(1-\theta)^2}\right|_{\theta=\bar{x}} = -\frac{n}{\bar{x}} - \frac{n}{1-\bar{x}} < 0$$

So $\bar{x}$ is indeed the MLE.

6.2.4 The likelihood function is given by $L(\theta \mid x_1, ..., x_n) = e^{-n\theta}\theta^{n\bar{x}}$, the log-likelihood function is given by $l(\theta \mid x_1, ..., x_n) = -n\theta + n\bar{x}\ln\theta$, and the score function is given by

$$S(\theta \mid x_1, ..., x_n) = -n + \frac{n\bar{x}}{\theta}.$$

Solving the score equation gives $\hat{\theta}(x_1, ..., x_n) = \bar{x}$. Note that since $\bar{x} \ge 0$, we have

$$\left.\frac{\partial S(\theta \mid x_1, ..., x_n)}{\partial \theta}\right|_{\theta=\bar{x}} = \left.-\frac{n\bar{x}}{\theta^2}\right|_{\theta=\bar{x}} = -\frac{n}{\bar{x}} < 0$$

so $\bar{x}$ is the MLE.

6.2.11 The parameter of the interest is changed to the volume $\eta = \mu^3$ from the length of a side μ. Then the likelihood function is also changed to

$$L_v(\eta|s) = L_v(\mu^3|s) = L_l(\mu|s)$$

where L_v is the likelihood function when the volume parameter $\eta = \mu^3$ is of the interest and L_l is the likelihood function of the length of a side parameter μ. The maximizer η of $L_v(\eta|s)$ is also a maximizer of $L_l(\eta^{1/3}|s)$. In other words, the MLE is invariant under 1-1 smooth parameter transformations. Hence, the MLE of η is equal to $\hat{\mu}^3 = (3.2\text{cm})^3 = 32.768\text{cm}^3$.