HW5 Solution

6.4.2 Recall that, the variance of a random variable can be expressed in terms of the moments as $\sigma_X^2 = \mu_2 - \mu_1^2$. Hence, the method of moments estimator of the population variance is given by $\hat{\sigma}_X^2 = m_2 - m_1^2$. To check if this estimator is unbiased we compute

$$E\left(m_2 - m_1^2\right) = \mu_2 - \left(\mathrm{Var}\left(m_1\right) + E^2\left(m_1\right)\right) = \mu_2 - \left(\frac{1}{n}\left(\mu_2 - \mu_1^2\right) + \mu_1^2\right)$$
$$= \left(1 - \frac{1}{n}\right)\sigma_X^2$$

Hence, this estimator is not unbiased.

6.4.3 The method of moments estimator of the coefficient of variation of a random variable X is $\sqrt{m_2 - m_1^2}/m_1$. Now let $Y = cX$. The $E(Y) = cE(X)$ and $\mathrm{Var}(Y) = c^2\,\mathrm{Var}(X)$. Therefore, the coefficient of variation of Y is

$$c\,\mathrm{Sd}\left(X\right)/cE(X) = \mathrm{Sd}\left(X\right)/E(X)$$

which is the coefficient of variation of X.

6.4.7 The empirical cdf is given by the following table. The sample median is estimated by $-.03$ and the first quartile is -1.28, while the third quartile is .98. The value $F(2)$ is estimated by $\hat{F}\left(2\right) = \hat{F}\left(1.36\right) = .90$.

i	$x_{(i)}$	$\hat{F}\left(x_{(i)}\right)$	i	$x_{(i)}$	$\hat{F}\left(x_{(i)}\right)$
1	-1.42	0.06	11	0.00	0.55
2	-1.35	0.10	12	0.38	0.60
3	-1.34	0.15	13	0.40	0.65
4	-1.29	0.20	14	0.44	0.70
5	-1.28	0.25	15	0.98	0.75
6	-1.02	0.30	16	1.06	0.80
7	-0.58	0.35	17	1.06	0.85
8	-0.35	0.40	18	1.36	0.90
9	-0.24	0.45	19	2.05	0.95
10	-0.03	0.50	20	2.13	1.00

7.1.3 First, the prior distribution of θ is $N(0, 10)$, therefore, the prior probability that θ is positive is 0.5. Next, the posterior distribution of θ is

$$N\left(\left(\frac{1}{10}+\frac{10}{1}\right)^{-1}\left(\frac{10}{1}\right), \left(\frac{1}{10}+\frac{10}{1}\right)^{-1}\right) = N\left(0.99010, 9.9010 \times 10^{-2}\right).$$

Therefore, the posterior probability that $\theta > 0$ is
$1-\Phi\left((0-0.99010)/\sqrt{9.9010 \times 10^{-2}}\right) = 1-\Phi(-3.1466) = 1-0.0008 = 0.9992.$

7.1.5 The likelihood function is given by $L(\theta \mid x_1, ...x_n) = \frac{1}{\theta^n} I_{[x_{(n)},\infty)}(\theta)$. The prior distribution is the same as in the previous exercise. The posterior distribution of θ is then given by

$$\pi(\theta \mid x_1, ...x_n) \propto \theta^{\alpha-n-1} e^{-\beta\theta} I_{[x_{(n)},\infty)}(\theta) \Big/ \int_{x_{(n)}}^{\infty} \theta^{\alpha-n-1} e^{-\beta\theta}\, d\theta.$$