STAT 516: Continuous random variables: probability density functions, cumulative density function, quantiles, and transformations

Lecture 6: Expectation of functions and Moments. Cauchy density

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Definition of Expectation

- Let X be a continuous random variable with a pdf $f(x)$. Let also $\int_{-\infty}^{\infty} |x| f(x)\,dx < \infty$. We say, then, that the expected value (mean) of X exists and is

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

- Let X be a continuous random variable with pdf $f(x)$. Let $g(X)$ be a function of X. The expectation of $g(X)$ exists if and only if $\int_{-\infty}^{\infty} |g(x)| f(x)\,dx < \infty$ in which case it is

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

Definition of Moments

- Let X be a continuous random variable with pdf $f(x)$. Then, the kth moment of X is defined as $E(X^k)$ for any $k \geq 1$. We say that the kth moment does not exist if $E(|X^k|) = \infty$.
- Let X be a continuous random variable with pdf $f(x)$. Suppose the expectation of X exists and let $\mu = E(X)$. Then, the variance of X is defined as $V(X) = \sigma^2 = E[(X - \mu)^2]$. Also, we say that the variance of X does not exist if $E[(X - \mu)^2] = \infty$.

Variance property

- Suppose X is a continuous random variable with the pdf $f(x)$. Then its variance, provided it exists, is equal to

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = E(X^2) - [E(X)]^2$$

- Suppose X has a symmetric distribution around some number a, i.e. $X - a$ and $a - X$ have the same distribution. Then, $E[(X-a)^{2k+1}] = 0$, for every $k \geq 0$, provided the expectation $E[(X-a)^{2k+1}]$ exists.

Examples of moment calculation

- Let $X \sim Unif[0,1]$. Check that
  1. $E(X^k) = \frac{1}{k+1}$
  2. $V(X) = E[(X - \frac{1}{2})^2] = \frac{1}{12}$

Example

- A lion sets a circular territory for itself by choosing a radius at random that is exponentially distributed (a unit is a mile). What is the expected area of the lion's territory?
- The radius $X \sim Exp(1)$; the area is πx^2 and so

$$E(area) = \int_0^\infty \pi x^2 e^{-x}\, x = \pi \int_0^\infty x^2 e^{-x}\, dx = 2\pi$$

Area of random triangle

- An equilateral triangle has the common side $X \sim Unif[0,1]$. What is the mean and the variance of the area of this triangle?
- If the sides are a, b, c, the area is $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$. If all sides are equal, area is $\frac{\sqrt{3}}{4}a^2$
- The mean is

$$E\,Y = \frac{\sqrt{3}}{4} E\,X^2 = \frac{1}{4\sqrt{3}}$$

- The variance is

$$Var\,(Y) = E\,Y^2 - (E\,Y)^2 = \frac{3}{16} E\,X^4 - \frac{1}{48} = \frac{1}{60}$$

Example

- In the "broken stick" ecological model, the proportion of the resource controlled by species 1 has the uniform distribution on $[0, 1]$
- The species that controls the majority of this resource controls the amount

$$h(X) = \max(X, 1 - X) = \begin{cases} 1 - X & 0 \leq X \leq \frac{1}{2} \\ X & \frac{1}{2} \leq X \leq 1 \end{cases}$$

- The expected amount controlled by the species having majority control is

$$E\,h(X) = \int_0^1 \max(x, 1 - x) * 1 d\,x = \frac{3}{4}$$

Example

- Let a horizontal line segment of length 5 be split into two parts at random
- The length of the left-hand part X has the pdf

$$f(x) = \begin{cases} \frac{1}{5} & 0 < x < 5 \\ 0 & \text{elsewhere} \end{cases}$$

- The expected length of X is $\mathbb{E}X = \frac{5}{2}$ and $\mathbb{E}(5 - X) = \frac{5}{2}$
- Note that

$$\mathbb{E}[(X(5-X)] = \mathbb{E}(5X - X^2) = \frac{25}{6} \neq \left(\frac{5}{2}\right)^2$$

Gamma function

- The gamma function is defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1}\, dx$$

  for any $\alpha > 0$
- Properties:
  1. $\Gamma(n) = (n-1)!$ for any positive integer n
  2. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)\ \forall \alpha > 0$
  3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Moments of the exponential distribution

- Let $X \sim Exp(1)$
- Check that $E(X^n) = \Gamma(n+1) = n!$ for any n...e.g. $EX = 1$ and $EX^2 = 2$, $Var(X) = 1$

Absolute value of a standard normal

- Let $X \sim N(0,1)$; find $E(|X|)$
- By definition,

$$E(|X|) = \int_{-\infty}^{\infty} |x|\phi(x)\,dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} xe^{-x^2/2}\,dx$$

$$= \sqrt{\frac{2}{\pi}}$$

Cauchy distribution

- Suppose a person holds a flashlight in his hand and stands one foot away from an infinitely long wall
- He points a beam of light in a random direction, i.e. the point where ray lands makes an angle X with an individual
- The individual is viewed as a straight line one foot long and $X \sim Unif[-\pi/2, \pi/2]$.
- Let Y be the horizontal distance from the person of the point at which the light lands
- Y is negative if the light lands on the person's left and positive if on the right

Cauchy distribution

- Clearly,

$$\tan(X) = \frac{Y}{1}$$

  and so $Y = \tan(X)$
- Note that $g(X) = \tan(X)$ is a strictly monotone function of X and $g^{-1}(y) = \arctan(y)$, $-\infty < y < \infty$,
- Finally, $g^{'}(x) = 1 + x^2$ and so

$$f_Y(y) = \frac{\frac{1}{\pi}}{1 + [\tan(\arctan(y))]^2} = \frac{1}{\pi(1 + y^2)}$$

  for $-\infty < y < \infty$
- Thus, Y has a standard Cauchy distribution

No finite expectation for Cauchy distribution!

- Let X be Cauchy distributed with $f(x) = \frac{1}{\pi(1+x^2)}$ $-\infty < x < \infty$
- Note that

$$E(|X|) = \int_{-\infty}^{\infty} |x| f(x)\, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|x|}{1+x^2}\, dx$$

$$\geq \frac{1}{\pi} \int_{0}^{\infty} \frac{x}{1+x^2}\, dx \geq \frac{1}{\pi} \int_{0}^{M} \frac{x}{1+x^2}\, dx$$

  for some $M > 0$

- Thus,

$$E(|X|) \geq \frac{1}{2\pi} \log(1 + M^2)$$

- Letting $M \to \infty$ we find that $E(|X|) = \infty$

Moments of the normal distribution

- First, take $k = 2n + 1$ for $n \geq 0$. Then,

$$E(X^k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n+1} e^{-x^2/2}\, dx = 0$$

- Now, take $k = 2n$ for $n \geq 1$. Then,

$$E(X^k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2}\, dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^{2n} e^{-x^2/2}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} z^{n-1} e^{-z/2}\, dx$$

with the substitution $z = x^2$.

Moments of the normal distribution

- Make another substitution: $u = z/2$. Then,

$$E(X^{2n}) = \frac{2^n}{\sqrt{\pi}} \int_0^\infty u^{n-1/2} e^{-u}\, du$$

and so

$$E(X^{2n}) = \frac{2^n \Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}}$$

- Using Gamma duplication formula $\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi} 2^{1-2n} \frac{(2n-1)!}{(n-1)!}$ we have

$$E(X^{2n}) = \frac{(2n)!}{2^n n!}$$