Lecture 6: Block Designs

Montgomery: Chapter 4

Nuisance Factor (may be present in experiment)

- Has effect on response but its effect is not of interest
- If unknown $\longrightarrow$ Protecting experiment through randomization
- If known (measurable) but uncontrollable $\longrightarrow$ Analysis of Covariance (Chapter 15 or 14 Section 3)
- If known and controllable $\longrightarrow$ Blocking

Penicillin Experiment

In this experiment, four penicillin manufacturing processes (A, B, C and D) were being investigated. Yield was the response. It was known that an important raw material, corn steep liquor, was quite variable. The experiment and its results were given below:

	blend 1	blend 2	blend 3	blend 4	blend 5
A	89_1	84_4	81_2	87_1	79_3
B	88_3	77_2	87_1	92_3	81_4
C	97_2	92_3	87_4	89_2	80_1
D	94_4	79_1	85_3	84_4	88_2

- Blend is a nuisance factor, treated as a block factor;
- (Complete) Blocking: all the treatments are applied within each block, and they are compared within blocks.
- Advantage: Eliminate blend-to-blend (between-block) variation from experimental error variance when comparing treatments.
- Cost: degree of freedom.

Randomized Complete Block Design

- b blocks each consisting of (partitioned into) a experimental units
- a treatments are randomly assigned to the experimental units within each block
- Typically after the runs in one block have been conducted, then move to another block.
- Typical blocking factors: day, batch of raw material etc.
- Results in restriction on randomization because randomization is only within blocks.
- Data within a block are dependent on each other. When $a = 2$, randomized complete block design becomes paired two sample case.

Statistical Model

- b blocks and a treatments
- Statistical model is

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

μ - grand mean

τ_i - ith treatment effect

β_j - jth block effect

$\epsilon_{ij} \sim \mathrm{N}(0, \sigma^2)$

- The model is additive because within a fixed block, the block effect is fixed; for a fixed treatment, the treatment effect is fixed across blocks. In other words, blocks and treatments do not interact.
- parameter constraints: $\sum_{i=1}^{a} \tau_i = 0; \qquad \sum_{j=1}^{b} \beta_j = 0$

Estimates for Parameters

- Rewrite observation y_{ij} as:

$$y_{ij} = \overline{y}_{..} + (\overline{y}_{i.} - \overline{y}_{..}) + (\overline{y}_{.j} - \overline{y}_{..}) + (y_{ij} - \overline{y}_{i.} - \overline{y}_{.j} + \overline{y}_{..})$$

- Compared with the model

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij},$$

- we have

$$\begin{aligned} \hat{\mu} &= \overline{y}_{..} \\ \hat{\tau}_i &= \overline{y}_{i.} - \overline{y}_{..} \\ \hat{\beta}_j &= \overline{y}_{.j} - \overline{y}_{..} \\ \hat{\epsilon}_{ij} &= y_{ij} - \overline{y}_{i.} - \overline{y}_{.j} + \overline{y}_{..} \end{aligned}$$

Sum of Squares (SS)

- Can partition $\mathrm{SS_T} = \sum\sum(y_{ij} - \overline{y}_{..})^2$ into

$$b\sum(\overline{y}_{i.} - \overline{y}_{..})^2 + a\sum(\overline{y}_{.j} - \overline{y}_{..})^2 + \sum\sum(y_{ij} - \overline{y}_{i.} - \overline{y}_{.j} + \overline{y}_{..})^2$$

$\mathrm{SS_{Treatment}}$	$= b\sum(\overline{y}_{i.} - \overline{y}_{..})^2$	$= b\sum\hat{\tau}_i^2$	df= $a-1$
$\mathrm{SS_{Block}}$	$= a\sum(\overline{y}_{.j} - \overline{y}_{..})^2$	$= a\sum\hat{\beta}_j^2$	df = $b-1$
$\mathrm{SS_E}$	$= \sum\sum(y_{ij} - \overline{y}_{i.} - \overline{y}_{.j} + \overline{y}_{..})^2$	$= \sum\sum\hat{\epsilon}_{ij}^2$	df = $(a-1)(b$

Hence:

- $\mathrm{SS_T} = \mathrm{SS_{Treatment}} + \mathrm{SS_{Block}} + \mathrm{SS_E}$
- The Mean Squares are
  $\mathrm{MS_{Treatment}} = \mathrm{SS_{Treatment}}/(a-1)$, $\mathrm{MS_{Block}} = \mathrm{SS_{Block}}/(b-1)$,
  and $\mathrm{MS_E} = \mathrm{SS_E}/(a-1)(b-1)$.

Testing Basic Hypotheses

- $H_0 : \tau_1 = \tau_2 = \ldots = \tau_a = 0$ vs H_1 : at least one is not
- Can show:

  E(MS$_{\mathrm{E}}$)=σ^2

  E(MS$_{\mathrm{Treatment}}$) = $\sigma^2 + b\sum_{i=1}^{a}\tau_i^2/(a-1)$

  E(MS$_{\mathrm{Block}}$) = $\sigma^2 + a\sum_{j=1}^{b}\beta_j^2/(b-1)$

- Use F-test to test H_0:

$$F_0 = \frac{\mathrm{MS}_{\mathrm{Treatment}}}{\mathrm{MS}_{\mathrm{E}}} = \frac{\mathrm{SS}_{\mathrm{Treatment}}/(a-1)}{\mathrm{SS}_{\mathrm{E}}/((a-1)(b-1))}$$

- Caution testing block effects
  - Usually not of interest.
  - Randomization is restricted: Differing opinions on F-test for testing blocking effects.
  - Can use ratio MS$_{\mathrm{Block}}$/MSE to check if blocking successful.
  - Block effects can be random effects. (considered fixed effects in this chapter)

Analysis of Variance Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Blocks	SS_{Block}	$b-1$	MS_{Block}	
Treatment	$\text{SS}_{\text{Treatment}}$	$a-1$	$\text{MS}_{\text{Treatment}}$	F_0
Error	SS_{E}	$(b-1)(a-1)$	MS_{E}	
Total	SS_{T}	$ab-1$		

$\text{SS}_{\text{T}} = \sum\sum y_{ij}^2 - y_{..}^2/N$

$\text{SS}_{\text{Treatment}} = \frac{1}{b}\sum y_{i.}^2 - y_{..}^2/N$

$\text{SS}_{\text{Block}} = \frac{1}{a}\sum y_{.j}^2 - y_{..}^2/N$

SS_{E}=SS_{T} - $\text{SS}_{\text{Treatment}}$- SS_{Block}

Decision Rule: If $F_0 > F_{\alpha,a-1,(b-1)(a-1)}$ then reject H_0

Example

An experiment was designed to study the performance of four different detergents in cleaning clothes. The following "cleanness" readings (higher=cleaner) were obtained with specially designed equipment for three different types of common stains. Is there a difference between the detergents?

	Stain 1	Stain 2	Stain 3
Detergent 1	45	43	51
Detergent 2	47	46	52
Detergent 3	48	50	55
Detergent 4	42	37	49

$$\sum\sum y_{ij} = 565 \text{ and } \sum\sum y_{ij}^2 = 26867$$

$y_{1.} = 139$, $y_{2.} = 145$, $y_{3.} = 153$ and $y_{4.} = 128$; $y_{.1} = 182$, $y_{.2} = 176$, and $y_{.3} = 207$

$\text{SS}_\text{T} = 26867 - 565^2/12 = 265$

$\text{SS}_\text{Trt} = (139^2 + 145^2 + 153^2 + 128^2)/3 - 565^2/12 = 111$

$\text{SS}_\text{Block} = (182^2 + 176^2 + 207^2)/4 - 565^2/12 = 135$

$\text{SS}_\text{E} = 265 - 111 - 135 = 19;\quad F_0 = (111/3)/(19/6) = 11.6$; P-value < 0.01

Checking Assumptions (Diagnostics)

- Assumptions
  - Model is additive (no interaction between treatment effects and block effects) (additivity assumption)
  - Errors are independent and normally distributed
  - Constant variance
- Checking normality:
  - Histogram, QQ plot of residuals, Shapiro-Wilk Test.
- Checking constant variance
  - Residual Plot: Residuals vs $\hat{y}_{ij}$
  - Residuals vs blocks
  - Residuals vs treatments

Checking Assumptions (Continued)

- Additivity
  - Residual Plot: residuals vs $\hat{y}_{ij}$
  - If residual plot shows curvilinear pattern, interaction between treatment and block likely exists
  - Interaction: block effects can be different for different treatments
- Formal test: Tukey's One-degree Freedom Test of Non-additivity
- If interaction exists, usually try transformation to eliminate interaction

Treatments Comparison

- Multiple Comparisons/Contrasts
  - procedures (methods) are similar to those for Completely Randomized Design (CRD)

    n is replaced by b in all formulas

    Degrees of freedom error is $(b-1)(a-1)$

- Example : Comparison of Detergents
  - Tukey's Method ($\alpha = .05$)

$$q_\alpha(a, df) = q_\alpha(4, 6) = 4.896.$$

$$CD = \frac{q_\alpha(4,6)}{\sqrt{2}}\sqrt{\text{MSE}(\frac{1}{b}+\frac{1}{b})} = 4.896\sqrt{\frac{19}{6*3}} = 5.001$$

Comparison of Treatment Means

Treatments

4	1	2	3
42.67	46.33	48.33	51.00
A	A		
	B	B	B

Using SAS

```
options nocenter ls=78;
goptions colors=(none);
symbol1 v=circle; axis1 offset=(5);

data wash;
 input stain soap y @@;
 cards;
1 1 45 1 2 47 1 3 48 1 4 42 2 1 43 2 2 46 2 3 50 2 4 37 3 1 51 3 2
52 3 3 55 3 4 49 ;

proc glm;
 class stain soap;
 model y = soap stain;
 means soap / alpha=0.05 tukey lines;
 output out=diag r=res p=pred;

proc univariate noprint normal;
 qqplot res / normal (L=1 mu=0 sigma=est);
 histogram res /normal (L=1 mu=0 sigma=est) kernel(L=2 K=quadratic);
```

```
run;

proc gplot;
 plot res*soap / haxis=axis1;
 plot res*stain / haxis=axis1;
 plot res*pred;
run;
```

Output

Dependent Variable: y

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	246.0833333	49.2166667	15.68	0.0022
Error	6	18.8333333	3.1388889		
Corrected Total	11	264.9166667			

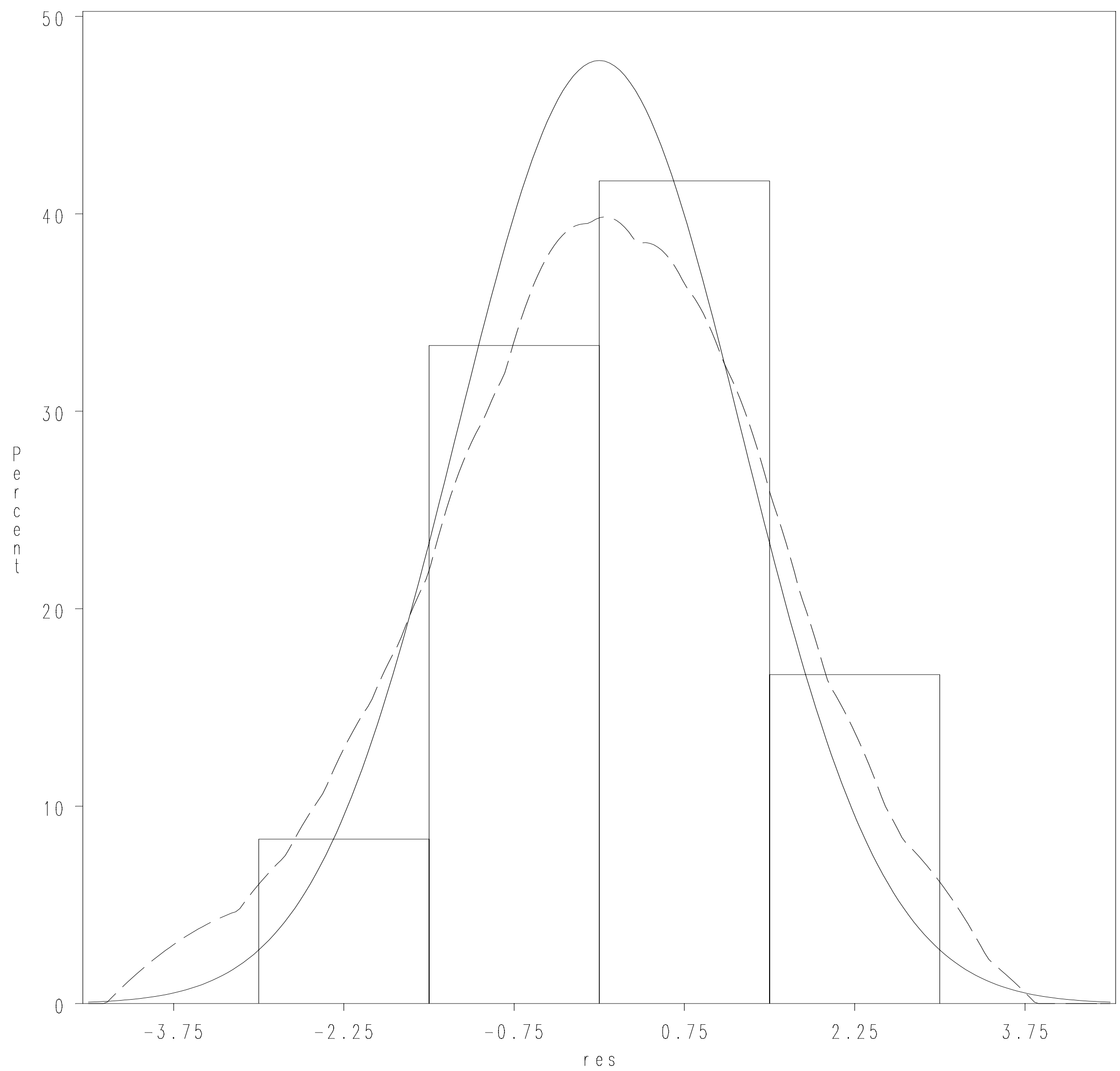
R-Square	Coeff Var	Root MSE	y Mean
0.928908	3.762883	1.771691	47.08333

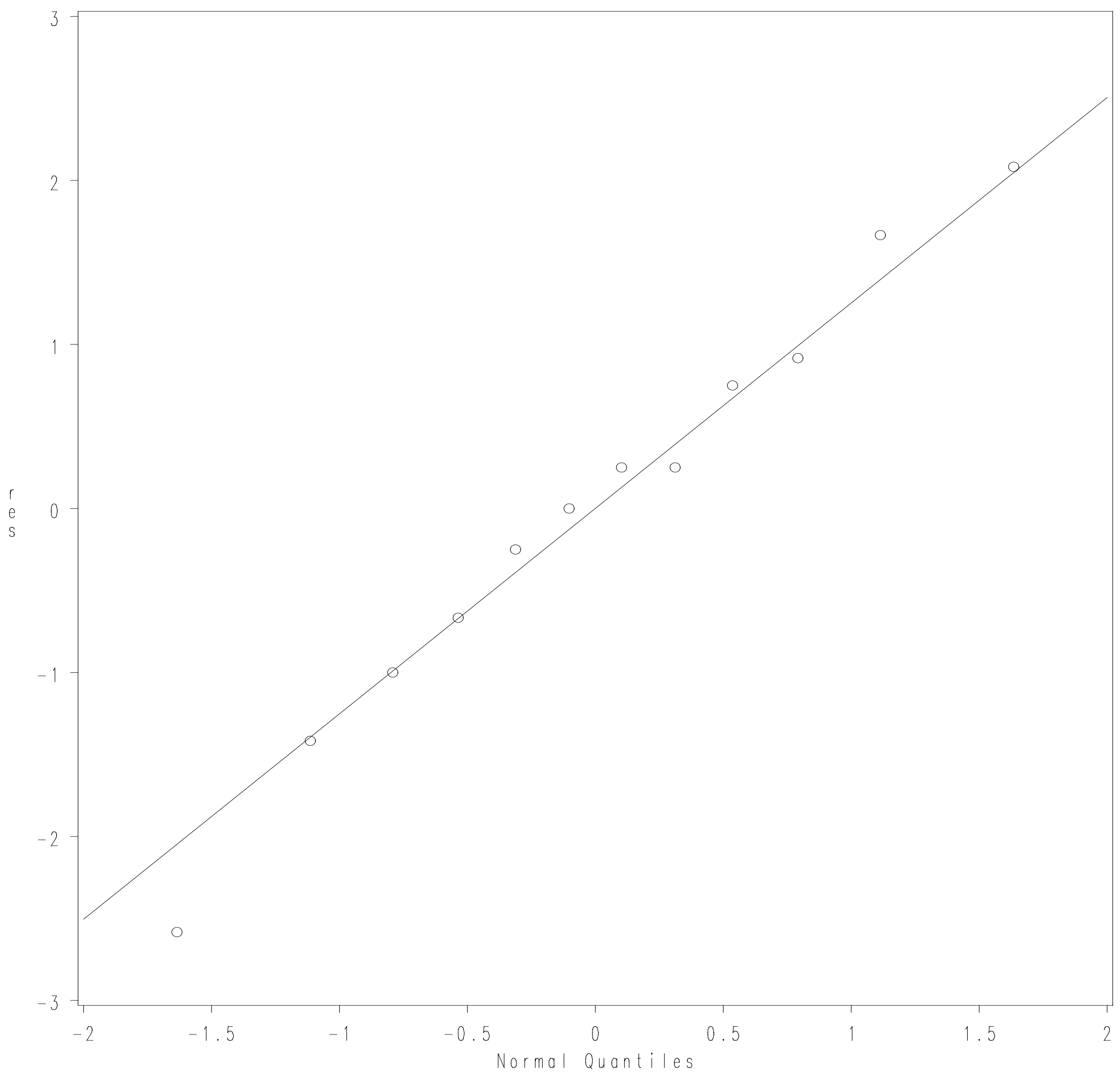
Source	DF	Type I SS	Mean Square	F Value	Pr > F
soap	3	110.9166667	36.9722222	11.78	0.0063
stain	2	135.1666667	67.5833333	21.53	0.0018

Source	DF	Type III SS	Mean Square	F Value	Pr > F
soap	3	110.9166667	36.9722222	11.78	0.0063
stain	2	135.1666667	67.5833333	21.53	0.0018

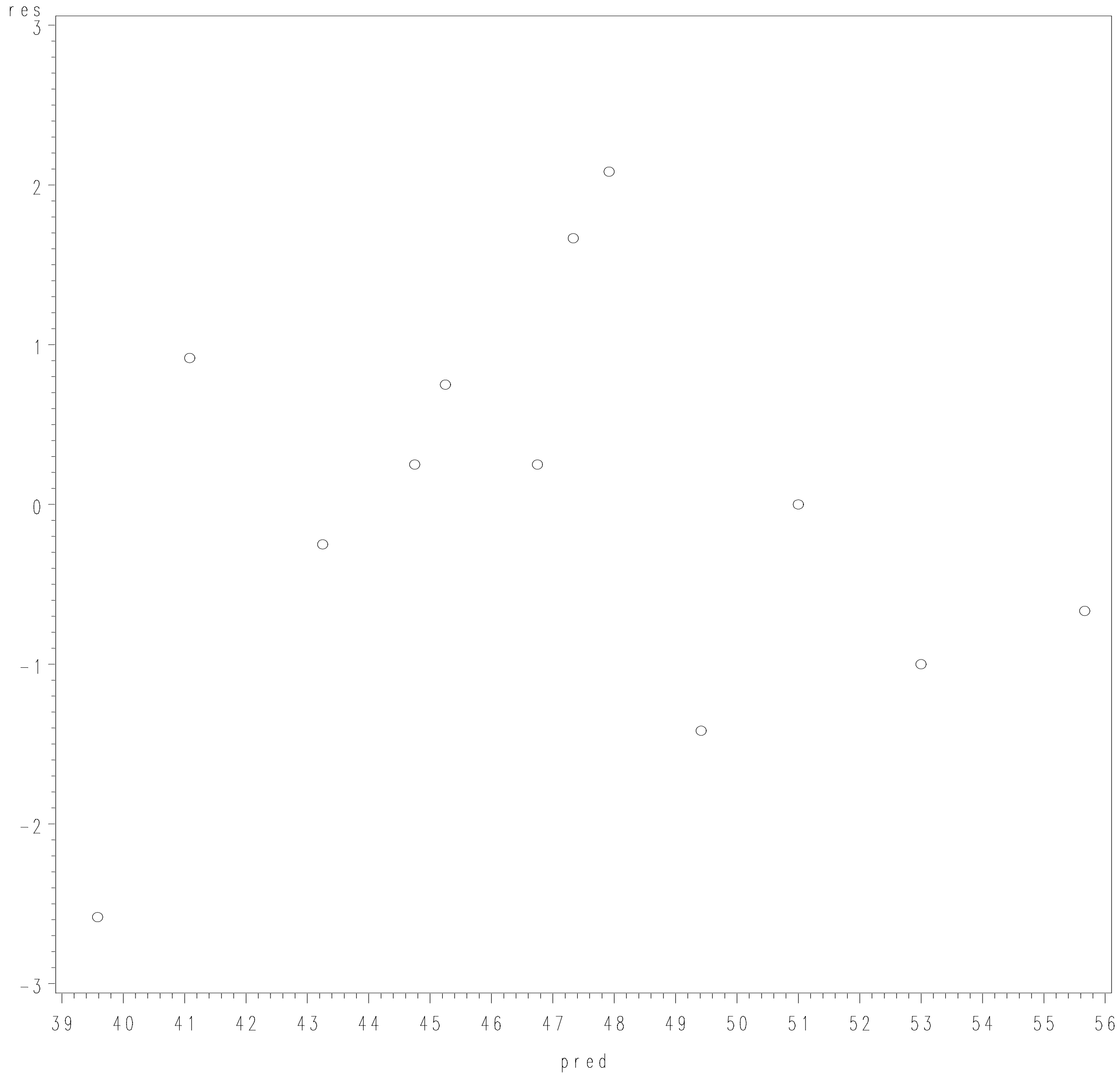
```
Tukey's Studentized Range (HSD) Test for res
      Alpha                                      0.05
      Error Degrees of Freedom                      6
      Error Mean Square                      3.138889
      Critical Value of Studentized Range   4.89559
      Minimum Significant Difference          5.007
 Means with the same letter are not significantly different.
         Tukey Grouping          Mean      N  soap

                     A         51.000      3    3
                     A
                     A         48.333      3    2
                     A
                B    A         46.333      3    1
                B
                B              42.667      3    4
```


50
40
30
20
10
0
Percent
-3.75
-2.25
-0.75
0.75
2.25
3.75
res



3
2
1
0
−1
−2
−3
r
e
s
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
Normal Quantiles


res
3
2
1
0
-1
-2
-3
39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56
pred

Tukey's Test for Non-additivity

- Additivity assumption (or no interaction assumption) is crucial for block designs or experiments.
- To check the interaction between block and treatment **fully** needs $(a-1)(b-1)$ degree of freedom. It is not affordable when without replicates.
- Instead consider a special type of interaction. Assume following model (pages 190-193 or pages 178-181)

$$y_{ij} = \mu + \tau_i + \beta_j + \gamma\tau_i\beta_j + \epsilon_{ij}$$

- $H_0 : \gamma = 0$ vs $H_1 : \gamma \neq 0$

Sum of Squares caused by possible interaction:

$$\text{SS}_N = \frac{\left[\sum_i \sum_j y_{ij} y_{i.} y_{.j} - y_{..}(\text{SS}_{\text{Trt}} + \text{SS}_{\text{Blk}} + y_{..}^2/ab)\right]^2}{ab\text{SS}_{\text{Trt}}\text{SS}_{\text{Blk}}} \quad df = 1.$$

Remaining error SS: $\text{SS}'_{\text{E}} = \text{SS}_{\text{E}} - \text{SS}_{\text{N}}, \quad df = (a-1)(b-1) - 1$

Test Statistic:

$$F_0 = \frac{\text{SS}_N/1}{\text{SS}'_{\text{E}}/[(a-1)(b-1)-1]} \sim F_{1,(a-1)(b-1)-1}$$

- Decision rule: Reject H_0 if $F_0 > F_{\alpha,1(a-1)(b-1)-1}$.

A Convenient Procedure to Calculate $\mathbf{SS_N}$, $\mathbf{SS'_E}$ and F_0

1 Fit additive model $y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$

2 Obtain $\hat{y}_{ij}$ and $q_{ij} = \hat{y}_{ij}^2$

3 Fit the model $y_{ij} = \mu + \tau_i + \beta_j + q_{ij} + \epsilon_{ij}$

Use the test for q_{ij} in the ANOVA table with type III SS and ignore the tests for the treatment and block factors.

Example 5-2 from Montgomery

- Impurity in chemical product is affected by temperature and pressure. We will assume temperature is a blocking factor. The data is shown below. We will test for non-additivity.

	Pressure				
Temp	25	30	35	40	45
100	5	4	6	3	5
125	3	1	4	2	3
150	1	1	3	1	2

$$\text{SS}_\text{N} = 0.0985, \text{SS}'_\text{E} = 1.9015, F_0 = .36, P - \text{value} = 0.566$$

Do Not Reject, there appears to be no interaction between block and treatment.

SAS Code

```
options nocenter ls=75;
data impurity;
input trt blk y @@;
cards;
1 1 5 1 2 3 1 3 1 2 1 4 2 2 1 2 3 1 3 1 6 3 2 4 3 3 3
4 1 3 4 2 2 4 3 1 5 1 5 5 2 3 5 3 2
;
proc glm;
class blk trt;
model y=blk trt;
output out=one r=res p=pred;

data two;
set one;
q=pred*pred;

proc glm data=two;
class blk trt;
model y=blk trt q/ss3; run;
```

Output

From the first model statement:

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	6	34.93333333	5.82222222	23.29	0.0001
Error	8	2.00000000	0.25000000		
Corrected Total	14	36.93333333			

Source	DF	Type I SS	Mean Square	F Value	Pr > F
blk	2	23.33333333	11.66666667	46.67	<.0001
trt	4	11.60000000	2.90000000	11.60	0.0021

Output

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	7	35.03185550	5.00455079	18.42	0.0005
Error	7	1.90147783	0.27163969		
Corrected Total	14	36.93333333			

Source	DF	Type III SS	Mean Square	F Value	Pr > F	
blk	2	1.25864083	0.62932041	2.32	0.1690	XXX
trt	4	1.09624963	0.27406241	1.01	0.4634	XXX
q	1	0.09852217	0.09852217	0.36	0.5660	

XXX: not meaningful for testing blocks and treatments

RCBD with Replicates

- a treatments ($i = 1, 2, \ldots, a$)
- b blocks ($j = 1, 2, \ldots, b$)
- n observations for each treatment in each block ($l = 1, 2, \ldots, n$)

$$y_{ijl} = \mu + \tau_i + \beta_j + \epsilon_{ijl} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ l = 1, 2, \ldots, n \end{cases}$$

- Similar assumptions as before. $N = abn$ and many more degree of freedom to get around. It allows interaction (but we have to be really careful about the possible interaction)

Assume no interaction between Ttr and Blk: ANOVA

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Blocks	SS$_{\mathrm{Block}}$	$b-1$	MS$_{\mathrm{Block}}$	
Treatment	SS$_{\mathrm{Treatment}}$	$a-1$	MS$_{\mathrm{Treatment}}$	F_0
Error	SS$_{\mathrm{E}}$	$abn-b-a+1$	MS$_{\mathrm{E}}$	
Total	SS$_{\mathrm{T}}$	$abn-1$		

For multiple comparison, df_E becomes $abn-a-b+1$ and the number of replicates for a fixed treatment now is bn instead of n. Hence, the formulas need to be modified accordingly.

Do not assume no interaction between Trt and Blk: ANOVA

- Assess additivity (no interaction) by Sum of Squares due to interaction $\text{SS}_{\text{Trt*Blk}}$.
- Interaction and error are not confounded; their SS's are separated

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Blocks	SS_{Block}	$b-1$	MS_{Block}	
Treatment	$\text{SS}_{\text{Treatment}}$	$a-1$	$\text{MS}_{\text{Treatment}}$	F_0
Blk*Trt	$\text{SS}_{\text{Blk}*\text{Trt}}$	$(b-1)(a-1)$	$\text{MS}_{\text{Blk}*\text{Trt}}$	
Error	SS_{E}	$ab(n-1)$	MS_{E}	
Total	SS_{T}	$abn-1$		

- Usually, we do not want to see large (significant) $\text{SS}_{\text{Blk}*\text{Trt}}$ because this will (1) affect the interpretation of the results and (2) the prediction of the fitted model. Sometimes, transformation is considered to eliminate interaction.