# Answer Keys to Homework#9

## Problem 1

(a) For the restricted model, we have the following EMS table.

| Term | F<br>a<br>i | F<br>b<br>j | R<br>c<br>k | R<br>n<br>l | EMS |
|---|---|---|---|---|---|
| $\tau_i$ | 0 | b | c | n | $\sigma^2 + bn\sigma^2_{\tau\gamma} + \frac{bcn\sum_{i=1}^a \tau_i^2}{a-1}$ |
| $\beta_j$ | a | 0 | c | n | $\sigma^2 + an\sigma^2_{\beta\gamma} + \frac{acn\sum_{j=1}^a \beta_j^2}{b-1}$ |
| $\gamma_k$ | a | b | 1 | n | $\sigma^2 + abn\sigma^2_{\gamma}$ |
| $(\tau\beta)_{ij}$ | 0 | 0 | c | n | $\sigma^2 + n\sigma^2_{\tau\beta\gamma} + \frac{cn\sum_{i=1}^a (\tau\beta)_{ij}^2}{(a-1)(b-1)}$ |
| $(\tau\gamma)_{ik}$ | 0 | b | 1 | n | $\sigma^2 + bn\sigma^2_{\tau\gamma}$ |
| $(\beta\gamma)_{jk}$ | a | 0 | 1 | n | $\sigma^2 + an\sigma^2_{\beta\gamma}$ |
| $(\tau\beta\gamma)_{ijk}$ | 0 | 0 | 1 | n | $\sigma^2 + n\sigma^2_{\tau\beta\gamma}$ |
| $\epsilon_{(ijk)l}$ | 1 | 1 | 1 | 1 | $\sigma^2$ |

The F-tests for the main effects are
Test '$H_0$: all $\tau_i$ are zero' vs. '$H_1$: at least one of $\tau_i$ is not zero' with $F_\tau = MSA/MSAC$, which follows $F_{a-1,(a-1)(c-1)}$ under the null hypothesis.
Test '$H_0$: all $\beta_j$ are zero' vs. '$H_1$: at least one of $\beta_j$ is not zero' with $F_\beta = MSB/MSBC$, which follows $F_{b-1,(b-1)(c-1)}$ under the null hypothesis.
Test '$H_0$: all $\gamma_k$ are zero' vs. '$H_1$: at least one of $\gamma_k$ is not zero' with $F_\gamma = MSC/MSE$, which follows $F_{c-1,abc(n-1)}$ under the null hypothesis.

(b) For the unrestricted model, we have the following EMS table.

| Term | F<br>a<br>i | F<br>b<br>j | R<br>c<br>k | R<br>n<br>l | EMS |
|---|---|---|---|---|---|
| $\tau_i$ | 0 | b | c | n | $\sigma^2 + bn\sigma^2_{\tau\gamma} + n\sigma^2_{\tau\beta\gamma} + \frac{bcn\sum_{i=1}^a \tau_i^2}{a-1}$ |
| $\beta_j$ | a | 0 | c | n | $\sigma^2 + an\sigma^2_{\beta\gamma} + n\sigma^2_{\tau\beta\gamma} + \frac{acn\sum_{j=1}^a \beta_j^2}{b-1}$ |
| $\gamma_k$ | a | b | 1 | n | $\sigma^2 + abn\sigma^2_{\gamma} + bn\sigma^2_{\tau\gamma} + an\sigma^2_{\beta\gamma} + n\sigma^2_{\tau\beta\gamma}$ |
| $(\tau\beta)_{ij}$ | 0 | 0 | c | n | $\sigma^2 + n\sigma^2_{\tau\beta\gamma} + \frac{cn\sum_{i=1}^a (\tau\beta)_{ij}^2}{(a-1)(b-1)}$ |
| $(\tau\gamma)_{ik}$ | 1 | b | 1 | n | $\sigma^2 + bn\sigma^2_{\tau\gamma} + n\sigma^2_{\tau\beta\gamma}$ |
| $(\beta\gamma)_{jk}$ | a | 1 | 1 | n | $\sigma^2 + an\sigma^2_{\beta\gamma} + n\sigma^2_{\tau\beta\gamma}$ |
| $(\tau\beta\gamma)_{ijk}$ | 1 | 1 | 1 | n | $\sigma^2 + n\sigma^2_{\tau\beta\gamma}$ |
| $\epsilon_{(ijk)l}$ | 1 | 1 | 1 | 1 | $\sigma^2$ |

The F-tests for the main effects are
Test '$H_0$: all $\tau_i$ are zero' vs. '$H_1$: at least one of $\tau_i$ is not zero' with $F_\tau = MSA/MSAC$, which follows $F_{a-1,(a-1)(c-1)}$ under the null hypothesis.
Test '$H_0$: all $\beta_j$ are zero' vs. '$H_1$: at least one of $\beta_j$ is not zero' with $F_\beta = MSB/MSBC$, which follows $F_{b-1,(b-1)(c-1)}$ under the null hypothesis.
Test '$H_0$: all $\gamma_k$ are zero' vs. '$H_1$: at least one of $\gamma_k$ is not zero' with $F_\gamma = (MSC + MSABC)/(MSAC +$

$MSBC$) (you may have a different approximate F-test here), which follows $F_{p,q)}$ under the null hypothesis with

$$\begin{aligned} p &= \frac{(MSC + MSABC)^2}{MSC^2/(c-1) + MSABC^2/((a-1)(b-1)(c-1))} \\ q &= \frac{(MSAC + MSBC)^2}{MSAC^2/((a-1)(c-1)) + MSBC^2/((b-1)(c-1))} \end{aligned}$$

## Problem 2

(a) The reason is: (i) "batch" has four levels at each level of "process"; (ii) under the same level of "process", the levels of "batch" are comparable; (iii) under a level of "process", the levels of "batch" can be arbitrarily numbered (i.e., the levels of "batch" from different levels of "process" are not comparable).

(b) Ignored.

## Problem 3

Note that certain variances terms are expected to be zero and thus dropped from the model. The EMS and F test are

| Sourse | EMS | Test |
|---|---|---|
| A | $bc\sigma_A^2 + c\sigma_{AB}^2 + \sigma^2$ | $\text{MS}_A/\text{MS}_{AB}$ |
| B | $ac\sigma_B^2 + c\sigma_{AB}^2 + a\sigma_{BC}^2 + \sigma^2$ | $(\text{MS}_B+\text{MS}_E)/(\text{MS}_{AB}+\text{MS}_{BC})$ |
| C | $ab\sigma_C^2 + a\sigma_{BC}^2 + \sigma^2$ | $\text{MS}_C/\text{MS}_{BC}$ |
| AB | $c\sigma_{AB}^2 + \sigma^2$ | $\text{MS}_{AB}/\text{MS}_E$ |
| BC | $a\sigma_{BC}^2 + \sigma^2$ | $\text{MS}_{BC}/\text{MS}_E$ |