Answer Keys to Homework#7

Problem 0

(a) I will answer this in terms of 4 trts rather than two sets of two treatments. A RCBD or BIBD could be used here. For the RCBD to be balanced, we utilize only four of the six possible runs each day and run the experiment over 6 days.

Source	DF
Day	5
Trt	3
Error	15
Total	23

For the BIBD, there are several ways to do this. If only two combinations are run each day, a total of 6 days are needed for the design to be balanced. Since $6 \times 2 = 12$, the combinations each day could be replicated or you could run the design over 12 days. Likewise, one could run three of the four treatment combinations each day. A total of 4 days are needed for the design to be balanced. Given there are six possible runs per day, each of these combinations could be replicated or run over a total of 8 days. Since more days takes away degrees of freedom from error, the replication design will be used. The ANOVA table below is for a BIBD with $k = 3$.

Source	DF
Day	3
Trt	3
Error	17
Total	23

(b) The std deviation of a treatment difference when using a RCBD is $\sqrt{2\sigma^2/b} = \sqrt{2\sigma^2/6}$. With 15 degrees of freedom, the t-statistic is 2.131. Thus the half-length of the confidence interval would be $2.131\sqrt{\sigma^2/3}$. For the BIBD above, the std deviation of a treatment difference is $\sqrt{2k\sigma^2/(\lambda a)} = \sqrt{2(3)\sigma^2/(4(4))}$. With 17 degrees of freedom the t statistic is 2.110 so the half length is $2.110\sqrt{3\sigma^2/8}$. In this situation,

$$\frac{2.131\sqrt{\sigma^2/3}}{2.110\sqrt{3\sigma^2/8}} < 1$$

so the RCBD is better. This is the best BIBD among the ones suggested so all other BIBDs are also worse. Other designs could be formed using only 4 days but they would not be balanced. If the experimenter was more interested in comparing certain treatment combinations then a RCBD could be combined with a PBIB over 4 days.

Problem 1

Consider a BIBD where $a = 4, b = 6, k = 2$.

Problem 2

(a) We have $a = 5$ treatments (gasoline additives) and $b = 5$ blocks (cars) with $a \leq b$. Each block contains $k = 4$ treatments, each treatment appears in $r = 4$ blocks, and each pair of treatments appears in the same blocks $\lambda = 3$ times. The total number of runs $N = ar = bk = 20$, and $\lambda(a-1) = r(k-1) = 12$. Hence this is a balanced incomplete block design.

(b) The ANOVA table from SAS is as follows (line for Model SS replaced by Type I SS of the block and the treatment)

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
car	4	31.20000000	7.80000000	8.57	0.0022
trt	4	35.73333333	8.93333333	9.81	0.0012
Error	11	10.01666667	0.91060606		
Corrected Total	19	76.95000000			

Since the p-value for treatment effect is very small $(= 0.0012)$, I conclude that there is a difference between the five gasoline additives.

(c) First, the row and column sums are calculated based on the original data table.

	car1	car2	car3	car4	car5	$y_{i\cdot}$
add1		17	14	13	12	56
add2	14	14		13	10	51
add3	12		13	12	9	46
add4	13	11	11	12		47
add5	11	12	10		8	41
$y_{\cdot j}$	50	54	48	50	39	

And the overall mean

$$\bar{y}_{\cdot\cdot} = \frac{1}{N}\sum_j y_{\cdot j} = \frac{50+54+48+50+39}{20} = 12.05.$$

Second, let's compute

$$Q_i = y_{i\cdot} - \frac{1}{k}\sum_j n_{ij} y_{\cdot j}, \qquad i = 1,2,3,4,5,$$

where n_{ij} equals 1 if treatment i appears in block j and 0 otherwise.

$$\begin{aligned}
Q_1 &= y_{1\cdot} - \frac{1}{4}\sum_{j=1}^{5} n_{1j} y_{\cdot j} = 56 - \frac{54+48+50+39}{4} = 8.25, \\
Q_2 &= y_{2\cdot} - \frac{1}{4}\sum_{j=1}^{5} n_{2j} y_{\cdot j} = 51 - \frac{50+54+50+39}{4} = 2.75, \\
Q_3 &= y_{3\cdot} - \frac{1}{4}\sum_{j=1}^{5} n_{3j} y_{\cdot j} = 46 - \frac{50+48+50+39}{4} = -0.75, \\
Q_4 &= y_{4\cdot} - \frac{1}{4}\sum_{j=1}^{5} n_{4j} y_{\cdot j} = 47 - \frac{50+54+48+50}{4} = -3.5, \\
Q_5 &= y_{5\cdot} - \frac{1}{4}\sum_{j=1}^{5} n_{5j} y_{\cdot j} = 41 - \frac{50+54+48+39}{4} = -6.75.
\end{aligned}$$

Finally, the adjusted means are calculated by

$$\hat{\mu}_i = \hat{\mu} + \hat{\tau}_i = \bar{y}_{\cdot\cdot} + \frac{kQ_i}{\lambda a}.$$

Hence

$$\begin{aligned}
\hat{\mu}_1 &= 12.05 + \frac{4 \cdot 8.25}{3 \cdot 5} = 14.25, \\
\hat{\mu}_2 &= 12.05 + \frac{4 \cdot 2.75}{3 \cdot 5} = 12.7833, \\
\hat{\mu}_3 &= 12.05 + \frac{4 \cdot (-0.75)}{3 \cdot 5} = 11.85, \\
\hat{\mu}_4 &= 12.05 + \frac{4 \cdot (-3.5)}{3 \cdot 5} = 11.1167, \\
\hat{\mu}_5 &= 12.05 + \frac{4 \cdot (-6.75)}{3 \cdot 5} = 10.25.
\end{aligned}$$

(d) The standard error of the difference between two treatment estimates is

$$\sqrt{\hat{Var}(\hat{\tau}_i - \hat{\tau}_j)} = \sqrt{\frac{2k}{\lambda a} MS_E} = \sqrt{\frac{2 \cdot 4}{3 \cdot 5} \cdot 0.9106} = 0.6969.$$

(e) The critical difference for Tukey's pairwise comparisons is

$$CD = \frac{q_{\alpha,a,ar-a-b+1}}{\sqrt{2}} \sqrt{\frac{2k}{\lambda a} MS_E} = \frac{q_{0.05,5,11}}{\sqrt{2}} \cdot 0.6969 = \frac{4.58}{\sqrt{2}} \cdot 0.6969 = 2.2569.$$

The five treatment mean estimates are ordered as $14.25 > 12.7833 > 11.85 > 11.1167 > 10.25$ ($\hat{\mu}_1 > \hat{\mu}_2 > \hat{\mu}_3 > \hat{\mu}_4 > \hat{\mu}_5$). After computing differences following this order and comparing them with the critical distance, I reach the following conclusion.

- The pairs of gasoline additives which have significantly different mileage performances are $(1,3), (1,4), (1,5), (2,5)$.
- The mileage performances within any other pairs of gasoline additives are not significantly different.

The results from SAS with the options "`lsmeans trt / tdiff adjust=tukey;`" are given below.

```
                 Least Squares Means
      Adjustment for Multiple Comparisons: Tukey-Kramer

                                   LSMEAN
             trt          y LSMEAN       Number
             1          14.2500000            1
             2          12.7833333            2
             3          11.8500000            3
             4          11.1166667            4
             5          10.2500000            5

             Least Squares Means for Effect trt
         t for H0: LSMean(i)=LSMean(j) / Pr > |t|

                   Dependent Variable: y
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i/j	1	2	3	4	5
1		2.104587	3.443869	4.496162	5.739781
		0.2838	0.0355	0.0065	0.0010
2	-2.10459		1.339282	2.391576	3.635195
	0.2838		0.6746	0.1884	0.0259
3	-3.44387	-1.33928		1.052293	2.295913
	0.0355	0.6746		0.8262	0.2167
4	-4.49616	-2.39158	-1.05229		1.243619
	0.0065	0.1884	0.8262		0.7280
5	-5.73978	-3.63519	-2.29591	-1.24362	
	0.0010	0.0259	0.2167	0.7280	

In the SAS output, the pairs that are significant different have p-values (the bottom values for entries in the output table) less than 0.05. Hence, the significantly different pairs from SAS are $(1,3),(1,4),(1,5),(2,5)$, which are consistent with my conclusion.

(f) The contrast I use is $\mathbf{C} = (1, 1, 0, -1, -1)'$. The significance test of it is shown below.

Contrast	DF	Contrast SS	Mean Square	F Value	Pr > F
C	1	30.10416667	30.10416667	33.06	0.0001

Since the p-value for the test is very small ($= 0.0001$), I conclude that the combination of additives 1 and 2 has significantly different characteristics from the combination of additives 4 and 5.