1. The testing booklet contains 12 questions.

2. Permitted Texas Instruments calculators:
   - BA-35
   - BA II Plus
   - BA II Plus Professional
   - TI-30Xa
   - TI-30X IIS (solar)
   - TI-30X IIB (battery)
   The memory of the calculator should be cleared at the start of the exam, if possible.

3. Mark your answers on the Scantron sheet using a #2 pencil.

4. Make sure to supply answers to all of the questions. There is no penalty for guessing.

5. No partial credit will be given.

6. Show all of your work in the exam booklet.

7. Extra sheets of paper are available from the instructor.
1. Suppose that 3 balls are chosen with replacement from an urn consisting of 6 white and 12 red balls. Let $X_i$ equal 1 if the $i$th ball selected is white, and let $X_i$ equal 0 otherwise. Give the joint probability mass function $p(x_1, x_2)$ of $X_1, X_2$.

A.) $p(0, 0) = 35/51$; $p(0, 1) = 5/34$; $p(1, 0) = 5/34$; $p(1, 1) = 1/51$

B.) $p(0, 0) = 22/51$; $p(0, 1) = 4/17$; $p(1, 0) = 4/17$; $p(1, 1) = 5/51$

C.) $p(0, 0) = 512/729$; $p(0, 1) = 817/5832$; $p(1, 0) = 817/5832$; $p(1, 1) = 17/972$

D.) $p(0, 0) = 4/9$; $p(0, 1) = 2/9$; $p(1, 0) = 2/9$; $p(1, 1) = 1/9$

E.) $p(0, 0) = 1/16$; $p(0, 1) = 3/16$; $p(1, 0) = 3/16$; $p(1, 1) = 9/16$

**Answer.** We see that $p(0, 0) = (12/18)^2 = 4/9$ is the probability of neither of the first two balls being white. Also $p(0, 1) = (12/18)(6/18) = 2/9$ is the probability that the second ball is white but the first ball is not white. Similarly, $p(1, 0) = 2/9$ is the probability that the first ball is white but the second ball is not white. Finally, $p(1, 1) = (6/18)^2 = 1/9$ is the probability that both of the first two balls are white.

So the correct answer is **D**, namely, $p(0, 0) = 4/9$; $p(0, 1) = 2/9$; $p(1, 0) = 2/9$; $p(1, 1) = 1/9$. 
2. The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} 2e^{-x-y} & 0 < y < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal densities of $X$ and $Y$.

A.) $f_X(x) = \begin{cases} 2e^{-x}(1 - e^{-x}) & x > 0 \\ 0 & \text{otherwise} \end{cases}$  
   $f_Y(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$

B.) $f_X(x) = \begin{cases} 2e^{-x}(1 - e^{-x}) & x > 0 \\ 0 & \text{otherwise} \end{cases}$  
   $f_Y(y) = \begin{cases} 2e^{-2y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$

C.) $f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$  
   $f_Y(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$

D.) $f_X(x) = \begin{cases} 2e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$  
   $f_Y(y) = \begin{cases} 2e^{-y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$

E.) $f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$  
   $f_Y(y) = \begin{cases} 2e^{-2y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$

**Answer.** For $x \leq 0$, we have $f_X(x) = 0$ since $X$ is never smaller than 0. Similarly, $f_Y(y) = 0$ for $y \leq 0$. Now we compute, for $x > 0$ and for $y > 0$, the marginal densities

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{0}^{x} 2e^{-x-y} \, dy = 2e^{-x}(1 - e^{-x})$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \int_{y}^{\infty} 2e^{-x-y} \, dx = 2e^{-2y}$$

So the correct answer is **B**, namely,

$$f_X(x) = \begin{cases} 2e^{-x}(1 - e^{-x}) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 2e^{-2y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$
3. The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} 9e^{-3x-3y} & x > 0, \ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[X + Y]$.

A.) 1/9  \quad B.) 1/6  \quad C.) 1/3  \quad D.) 2/3  \quad E.) 1

**Answer.** Perhaps the easiest method is to recognize that $X$ and $Y$ are independent; each are exponential random variables with parameter $\lambda = 3$, and thus $E[X] = 1/3$ and $E[Y] = 1/3$. So $E[X + Y] = E[X] + E[Y] = 2/3$.

Another method is to compute directly

$$E[X + Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y)f(x, y) \, dx \, dy = \int_{0}^{\infty} \int_{0}^{\infty} (x + y)(9e^{-3x-3y}) \, dx \, dy = 2/3$$

So the correct answer is D, namely, 2/3.
4. A man and woman are each located uniformly on a straight road of length 10. Assume that their locations are independent and uniformly distributed along the road. [That is, the distance from one of the fixed ends of the road is uniformly distributed over (0, 10).] Let $X$ denote the distance from the man to the woman. Find the density of $X$.

A.) $f_X(x) = \begin{cases} \frac{1}{100} & 0 < x < 10 \\ 0 & \text{otherwise} \end{cases}$

B.) $f_X(x) = \begin{cases} \frac{x}{5} - \frac{x^2}{100} & 0 < x < 10 \\ 0 & \text{otherwise} \end{cases}$

C.) $f_X(x) = \begin{cases} \frac{1}{10} & 0 < x < 10 \\ 0 & \text{otherwise} \end{cases}$

D.) $f_X(x) = \begin{cases} \frac{x}{5} & 0 < x < 10 \\ 0 & \text{otherwise} \end{cases}$

E.) $f_X(x) = \begin{cases} \frac{1}{5} - \frac{x}{50} & 0 < x < 10 \\ 0 & \text{otherwise} \end{cases}$

**Answer.** Finding $F_X(a)$ in this problem, by the way, is exactly the same as Problem 14 from Chapter 6, which was on the homework. I made the problem even easier by using a concrete number “10” instead of the abstract number “$L$”.

We see that, of course, $f_X(x) = 0$ for $x \leq 0$ and for $x \geq 10$.

For $0 < x < 10$, the cumulative distribution function of $X$ is easily computed by drawing a picture and finding the area of the shaded region divided by the area of the entire sample space. This yields $F_X(a) = P(X \leq a) = 1 - P(X > a) = 1 - \frac{1}{100}(10-a)^2 = \frac{a}{5} - \frac{a^2}{100}$.

Alternatively, we could compute

$$F_X(a) = P(X \leq a) = \int_0^a \int_{x-a}^{x+a} \frac{1}{100} \, dy \, dx$$

Either way, we have $F_X(x) = \frac{x}{5} - \frac{x^2}{100}$. Differentiating with respect to $x$ yields the density, $f_X(x) = \frac{1}{5} - \frac{x}{50}$.

So the correct answer is **E**, namely, $f_X(x) = \begin{cases} \frac{1}{5} - \frac{x}{50} & 0 < x < 10 \\ 0 & \text{otherwise} \end{cases}$
5. The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} \frac{3}{2}xy & 0 \leq x \leq 2, \ 0 \leq y \leq 2, \ 0 \leq x + y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[X]$.

A.) 2/5  B.) 8/15  C.) 2/3  D.) 4/5  E.) 1

**Answer.** We compute directly

$$E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) \, dy \, dx = \int_{0}^{2} \int_{0}^{2-x} (x) \left( \frac{3}{2}xy \right) \, dy \, dx = \frac{4}{5}$$

An alternate method is to compute directly

$$E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) \, dx \, dy = \int_{0}^{2} \int_{0}^{2-y} (x) \left( \frac{3}{2}xy \right) \, dx \, dy = \frac{4}{5}$$

So the correct answer is D, namely, 4/5.
6. If $X$ is uniformly distributed over $(0, 3)$ and $Y$ is exponentially distributed with parameter $\lambda = 1$, find $P(X + Y \leq 2)$. Assume that $X$ and $Y$ are independent.

A.) $\frac{2}{3}$  
B.) $\frac{1}{3}e^{-2} + 1 - \frac{1}{3}e$  
C.) $\frac{1}{3}e^{-3} + \frac{2}{3}$  
D.) $\frac{1}{3}e^{-2} + \frac{1}{3}$  
E.) $1 - e^{-2}$

**Answer.** The joint density of $X$ and $Y$ is

$$f(x, y) = \begin{cases} \frac{1}{3}e^{-y} & 0 < x < 3, \ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

We compute directly

$$P(X + Y \leq 2) = \int_0^2 \int_0^{2-x} \frac{1}{3}e^{-y} \ dy \ dx = \frac{1}{3}e^{-2} + \frac{1}{3}$$

An alternate method is to compute directly

$$P(X + Y \leq 2) = \int_0^2 \int_0^{2-y} \frac{1}{3}e^{-y} \ dx \ dy = \frac{1}{3}e^{-2} + \frac{1}{3}$$

So the correct answer is D, namely, $\frac{1}{3}e^{-2} + \frac{1}{3}$. 


A survey of citizens over age 25 in Lafayette and West Lafayette is taken. It shows that 69.7% of West Lafayette’s residents have a college degree. Also, 23.7% of Lafayette’s residents have a college degree. Suppose that random samples of 200 West Lafayette residents and 200 Lafayette residents are chosen. Approximate the probability that at least 196 of these 400 people have a college degree. [Hint: use continuity correction.]

A.) .16  B.) .40  C.) .46  D.) .54  E.) .84

**Answer.** We notice that the number of West Lafayette residents with a college degree, denoted by \(X\), is a Binomial with \(n = 200\) and \(p = 69.7\). The number of Lafayette residents with a college degree, denoted by \(Y\), is a Binomial with \(n = 200\) and \(p = 23.7\). So \(X + Y\) is a normal random variable with mean \(200 \times 0.697 + 200 \times 0.237 = 186.8\) and variance \(200 \times 0.697 \times (1 - 0.697) + 200 \times 0.237 \times (1 - 0.237) = 78.4044\). So

\[
P(X + Y \geq 196) = P(X + Y \geq 195.5) = P \left( \frac{X + Y - 186.8}{\sqrt{78.4044}} \geq \frac{195.5 - 186.8}{\sqrt{78.4044}} \right) = P(Z \geq 0.9825) = 1 - P(Z \leq 0.9825) = 1 - \Phi(0.98) \approx 1 - 0.8365 = 0.1635
\]

So the correct answer is **A**, namely, .16.
8. The random variables $X$ and $Y$ have constant joint density $f(x, y)$ on the region $x \geq 1$ and $0 \leq y \leq e^{-x}$; in other words

$$f(x, y) = \begin{cases} e & x \geq 1, \ 0 \leq y \leq e^{-x} \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional density $f_{Y|X}(y|x)$ for a fixed $X = x \geq 1$.

A.) $f_{Y|X}(y|x) = \begin{cases} e^x & 0 \leq y \leq e^{-x} \\ 0 & \text{otherwise} \end{cases}$

B.) $f_{Y|X}(y|x) = \begin{cases} e^{-x} & 0 \leq y \leq e^{-x} \\ 0 & \text{otherwise} \end{cases}$

C.) $f_{Y|X}(y|x) = \begin{cases} e & 0 \leq y \leq e^{-x} \\ 0 & \text{otherwise} \end{cases}$

D.) $f_{Y|X}(y|x) = \begin{cases} e^{1-x} & 0 \leq y \leq e^{-x} \\ 0 & \text{otherwise} \end{cases}$

E.) $f_{Y|X}(y|x) = \begin{cases} e^{-y} & 0 \leq y \leq e^{-x} \\ 0 & \text{otherwise} \end{cases}$

**Answer.** For $x \geq 1$, we see that $X$ has marginal density

$$f_X(x) = \int_0^{e^{-x}} e \, dy = e^{1-x}$$

so the conditional distribution is

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{e}{e^{1-x}} = e^x$$

So the correct answer is \textbf{A}, namely, $f_{Y|X}(y|x) = \begin{cases} e^x & 0 \leq y \leq e^{-x} \\ 0 & \text{otherwise} \end{cases}$.
9. Three people, Alice, Bob, and Carol, are each waiting for their spouses to arrive. The waiting times (in minutes) are independent exponential random variables, each with parameter $\lambda = 5$. Find the probability that no one waits more than 10 minutes.

A.) $1 - e^{-50}$  
B.) $1 - e^{-150}$  
C.) $e^{-50}$  
D.) $e^{-150}$  
E.) $(1 - e^{-50})^3$

**Answer.** Let $X, Y, Z$ denote Alice, Bob, and Carol’s waiting times, respectively. Then the desired probability is

$$P(X, Y, Z \leq 10) = P(X \leq 10)P(Y \leq 10)P(Z \leq 10) = (1 - e^{-(5)(10)})^3 = (1 - e^{-50})^3$$

So the correct answer is **E**, namely, $(1 - e^{-50})^3$. 


10. The random variables $X$ and $Y$ are independent and identically distributed random variables, each uniform on $(0, 1)$. Compute the joint density of $U = X$ and $V = X/Y$.

A.) $f(u, v) = \frac{v^2}{u}$  
B.) $f(u, v) = u/v$  
C.) $f(u, v) = \frac{u}{v^2}$  
D.) $f(u, v) = v/u$  
E.) $f(u, v) = u$

**Answer.** We note that $u = g_1(x, y) = x$ and $v = g_2(x, y) = x/y$. Also, we can retrieve that $x$’s and $y$’s from the $u$’s and $v$’s by writing $x = h_1(u, v) = u$ and $y = h_2(u, v) = u/v$. The Jacobian is

$$J(x, y) = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -\frac{x}{y^2}$$

and thus $|J|^{-1} = \frac{y^2}{x}$. Also, the joint density of $X$ and $Y$ is simply “1” because $X$ and $Y$ are independent and are each uniform on $(0, 1)$. So we conclude that the joint density of $U$ and $V$ is $f_{U, V}(u, v) = f_{X, Y}(x, y)|J|^{-1} = \frac{y^2}{x} = \frac{(u/v)^2}{u} = \frac{u}{v^2}$.

So the correct answer is C, namely, $f(u, v) = \frac{u}{v^2}$. 
11. A total of \( N \) people arrive separately to a professional dinner. Upon arrival, each person looks to see if he or she has any friends among those present. That person then either sits at the table of a friend or at an unoccupied table if none of those present is a friend. Assuming that each of the \( \binom{N}{2} \) pairs of people are, independently, friends with probability \( \frac{1}{2} \), find the expected number of occupied tables. [Hint: Let \( X_i \) equal 1 or 0 depending on whether the \( i \)th arrival sits at a previously unoccupied table.]

A.) \( (N)(2^{N-1}) \)  B.) \( 1 - 2^N \)  C.) \( (N)(2^{1-N}) \)  D.) \( 1 - 2^{-N} \)  E.) \( 2 - 2^{1-N} \)

Answer. We let \( X \) denote the number of occupied tables. As in the hint, we define

\[
X_i = \begin{cases} 
1 & \text{if the } i\text{th arrival sits at a previously unoccupied table} \\
0 & \text{otherwise}
\end{cases}
\]

and thus \( X = X_1 + \cdots + X_N \). So \( E[X] = E[X_1 + \cdots + X_N] = E[X_1] + \cdots + E[X_N] \). Since \( X_i \) only takes on the values 0 and 1, then \( E[X_i] = P(X_i = 1) \), which is the probability that the \( i \)th arrival sits a previously unoccupied table. This happens if and only if the \( i \)th arrival is not friends with any of the first \( i - 1 \) arrivals, so

\[
E[X_i] = P(X_i = 1) = \left( 1 - \frac{1}{2} \right)^{i-1} = \frac{1}{2^{i-1}}
\]

Thus

\[
E[X] = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{N-1}} = 1 + 1 - \frac{1}{2^{N-1}} = 2 - 2^{1-N}
\]

So the correct answer is **E**, namely, \( 2 - 2^{1-N} \).
12. An urn has \( m \) black balls. At each stage a black ball is removed and a new ball, that is black with probability \( p \) and white with probability \( 1 - p \), is put in its place. Find the expected number of stages needed until there are no more black balls in the urn.

A.) \( \frac{m}{1-p} \)  
B.) \( \frac{m}{p} \)  
C.) \( m(1 - p) \)  
D.) \( mp \)  
E.) \( pm^2 \)

**Answer.** We let \( X \) denote the number of stages needed until there are no more black balls in the urn. We define \( X_1 \) as the number of stages until the first black ball is replaced with a white ball. We define \( X_2 \) as the number of additional stages until the second black ball is replaced with a white ball. We define \( X_3 \) as the number of additional stages until the third black ball is replaced with a white ball. We define \( X_4, \ldots, X_m \) similarly. Thus \( X = X_1 + \cdots + X_m \). So \( E[X] = E[X_1 + \cdots + X_m] = E[X_1] + \cdots + E[X_m] \). We note that the \( X_i \)'s are each geometric random variables with \( 1 - p \) as the probability of “success”, i.e., with probability of \( 1 - p \) of replacing a black ball by a white ball. So \( E[X_i] = \frac{1}{1-p} \) for each \( i \).

Thus

\[
E[X] = \frac{1}{1-p} + \frac{1}{1-p} + \cdots + \frac{1}{1-p} = \frac{m}{1-p}
\]

So the correct answer is **A**, namely, \( \frac{m}{1-p} \).