Optimal consumption and annuity equivalent wealth with mortality model uncertainty

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Abstract

The classical Yaari (1965) lifecycle model (LCM) lies at the very heart of much modern retirement research generally, and the economic understanding of annuity demand particularly. The LCM predicts a high annuity demand among individuals facing retirement, yet it is rarely the case in reality. The disconnection between economic theory and practice—widely known as the annuity puzzle—has spurred intensified research attempting to demystify the economic and behavioral underpinnings.

In this paper, we examine the cause of low annuity demand through the angle of mortality model uncertainty. To this end, we advance Yaari's LCM via incorporating with a mortality perturbation analysis. We obtain the optimal robust consumption rule and the annuity equivalent wealth. We find that investors may understate the incremental utility gained by annuitization if mortality model uncertainty is disregarded.

Keywords and phrases: Annuity puzzle, complete annuity market, complete bond market, model risk, perturbation analysis.

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1 Introduction

Owing to the growing public concern on retirement funding inadequacy, retirement planning has become a very active research area during recent decades. A constant focus has been placed on studying about how retirees should wisely drawdown their retirement nest eggs in order to maintain the standard of living in retirement. Toward this aim, researchers resort to the rational economic theory and seek the optimal blueprint for guiding retirees' saving and consumption behaviors. Originally postulated in Fisher (1930) under the assumption of deterministic time horizon and then refined by Yaari (1965) to a stochastic lifetime, the lifecycle model (LCM) of consumption has evolved as the building block of much modern retirement research. Namely, Yaari (1965) derived the optimal consumption rule for a utility-maximizing retiree who has no bequest motive and faces a stochastic time of death. Under an additively separable utility function, Yaari's (1965) analysis suggested that a rational investor should convert all the savings into an actuarially fair annuity upon retirement. Later on, rigorous analysis by Davidoff et al. (2005) showed that Yaari's (1965) conclusion remains valid even when most of the economic assumptions are relaxed.

Though economic theory predicts a high annuity demand, this is rarely the case in reality. Very few consumers facing retirement choose to annuitize a substantial portion of their retirement savings (Benartzi et al., 2011). This disparity between theory and the actual consumers' behavior, commonly referred to as the *annuity puzzle*, has spurred intensified research attempting to demystify the economic and behavioral underpinnings. Several explanations of the annuity puzzle have been proposed, which include low retirement savings amongst the population (Dushi and Webb, 2004), decreased asset liquidity (Pang and Warshawsky, 2010; Peijnenburg et al., 2017), lack of bequest motive (Lockwood, 2012), incomplete annuity market (Horneff et al., 2008; Koijen et al., 2011), unfair annuity pricing (Mitchell et al., 1999), and default risk of the annuity providers (Agnew et al., 2008), to name but only a few. It is fair to argue that none of the existing explanations have been shown to fully account for the low annuity demand in reality. However, the aforementioned studies together essentially benefit us to better understand the issue from different angles.

This paper bears another effort to unravel the annuity puzzle via the angle of mortality model uncertainty. With all the other complexities involved in retirement planning, the assessment of retiree's future mortality pathway plays a decisive role in the decision-making process. Unexpected deviations of the actual mortality evolution from the mortality model's prediction may pose a substantial influence on the lifespan discounted utility, turning the original model implied optimal strategy to be inferior. Nevertheless, modeling the individual mortality is notoriously hard from the statistical standpoint. Different than the objective mortality model which can be estimated from the population data, the micro-structure of the subjective mortality is extremely complicated and is closely related to the retiree's occupation, wealth, life style, and other socioeconomic determinants (Hurd and McGarry, 1995, 2002). To develop an effective retirement strategy, the subjective mortality model should be "best-estimated" using available data, while we also have to be mindful of the model risk associated with the best-estimated model.

In this paper, we treat the uncertainty surrounding the subjective mortality model as a robust control problem. That is, in addition to the best-estimated reference mortality model, we should consider an alternative set of statistically similar mortality models, among which we solve the retirement planning problem based on the so-called endogenous worst-case mortality scenario. Consequently, the consumption strategy obtained in our study will remain desirable even when the best-estimated mortality model performs inadequately. As a side note, in behavioral economics, an investor's fear of the uncertainty in the estimated probability distributions of future outcomes, is referred to as ambiguity aversion. In the context of this current paper, mortality ambiguity aversion represents a retiree's concern about the mortality model's misspecification. If a retiree has no mortality ambiguity aversion, that means the retiree would ignore the uncertainty surrounding the mortality model and fully trust the best-estimated mortality curve.

1.1 Summary of the contributions and findings

Our paper makes both technical and economic contributions. In terms of the technical contributions, we extend the classical Yaari's LCM in two innovative aspects. First, we integrate a perturbation analysis into the study of Yaari's LCM and obtain the optimal consumption policy that is robust to mortality model uncertainty. Both the complete annuity market and complete bond market are considered, and under the endogenous worst-case mortality scenario, we calculate the annuity equivalence wealth (AEW) which quantifies the incremental welfare gained by annuitization. Second, we generalize the additive utility considered in Yaari's LCM to the more general Epstein-Zin recursive utility (Duffie and Epstein, 1992). Particularly, recent retirement studies based on Yaari's LCM often assume the constant relative risk aversion (CRRA) utility. It is known that the CRRA utility restricts the risk aversion parameter to be the reciprocal of the elasticity of intertemporal substitution (EIS) parameter. However, these two parameters characterize very distinct features of retiree's preferences. The adoption of recursive utility in our study allows us to distinguish the coefficient of relative risk aversion from the EIS parameter. It will be interesting to study the roles of these two different risk preference parameters that play in determining the retiree's perception about mortality model uncertainty, the optimal consumption policy, and AEW.

Capitalizing on the extended LCM described above, we obtain the following economic findings. First, we discover that compared with the best-estimated mortality model, the worst-case mortality scenario obtained in the robust control analysis can be a deteriorate or improved mortality trajectory, depending on the value of the EIS parameter. If the EIS coefficient is smaller than one, which is prevalent as suggested by Yogo (2004), then the worst-case mortality scenario corresponds to an improved mortality trajectory, or equivalently a longevity risk scenario in retirement planning. This is a rather non-trivial yet appealing finding. It reveals that the recent retirement literature focusing on longevity risk is also meaningful from the mortality model uncertainty standpoint. Moreover, the worst-case perturbed mortality model is a proportional shift of the best-estimated reference mortality curve. This type of transform, also known as the proportional hazard distortion in Wang (1996), is often used in actuarial practice to examine the sensitivity of mortality assumptions, perhaps mainly because of its inherent simplicity. As a by-product, our study shows that the choice of proportional shock in sensitivity analysis is indeed sufficiently conservative for covering the worst-case mortality scenario in the sense of mortality model uncertainty. Second, by comparing the optimal consumption policies between retirees with and without mortality ambiguity aversion, we show rigorously that the presence of mortality ambiguity aversion increases the value of AEW for retirees having EIS coefficient smaller than one. In other words, if mortality model uncertainty is ignored, then the actual economic welfare gained by annuitization may be understated. We believe that this is a new behavioral explanation to the enduring economic puzzle on low annuity demand.

We note that another similar stream of recent studies on optimal annuity (and tontine) demand attack the mortality model uncertainty problem via mortality shocks that obey a parametric law (e.g., Chen et al., 2019, 2021, 2020). Specifically, they assume that the true state of the mortality model is shocked by a known random variable (RV) which is exogenous in the annuity demand problem and has a prior probability distribution. Another application of this mortality shock approach is for pricing mortality-linked securities (Chen et al., 2019). The robust control approach adopted by our paper contains mortality/longevity shocks that are determined endogeneously by a perturbation analysis, thus it is strikingly different from the aforementioned mortality shock approach.

It is noteworthy that Shen and Su (2019) studied the implications of models' uncertainty in a life-cycle asset allocation problem. However, this current paper should not be viewed as just a special case of Shen and Su (2019), and they are different in the following three major aspects. First, the focuses of the two papers are different. Namely, Shen and Su (2019) aimed to study the life-cycle optimal consumption-investment-insurance strategies under economics and mortality models uncertainty. To this end, they considered a substantially more complicated asset allocation model in order to mimic the real life actuarial and economic environments as realistic as possible. As such, intensive numerical analysis must be adopted to study the associated economic implications. In this paper, we aim to understand the implications of mortality model uncertainty on the annuity puzzle. Hence, we focus on Yaari's LCM which originally gave rise to the annuity puzzle. The inherent elegance of Yaari's LCM greatly benefits us to establish the relationship between mortality ambiguity aversion and annuity value. Almost all the results we obtain in this paper are explicit and very intuitive to interpret. Second, we calculate the AEW based on the proposed extended LCM, which promotes the major argument that we aim to convey via this paper. There was no investigation related to the AEW in Shen and Su (2019). Third, Shen and Su (2019) assumed the CRRA utility and showed that the worst-case mortality scenario is an improved (resp. deteriorated) mortality case if the relative risk aversion parameter is smaller (resp. greater) than one. In this paper, we consider the more general Epstein-Zin recursive utility. In so doing, we discover that the determinant of the worst-case mortality scenario is the EIS parameter but not the relative risk aversion parameter.

The annuity puzzle under ambiguous life expectancy was studied in Han and Hung (2021) based on the robust control approach. Their mathematical framework is similar to that of Shen and Su (2019), so our current paper is different from Han and Hung (2021) in at least the same aspects mentioned in the previous paragraph. In addition, instead of the optimal annuity purchase considered in Han and Hung (2021), our current paper attacks the annuity puzzle problem via the notion of AEW which has the merit of conciseness and transparency in communicating the utility increment coming from annuitization (Bernard et al., 2021; Milevsky and Huang, 2018). Further, the utility function considered in Han and Hung (2021) is restricted to be positive, while we consider both positive and negative utility function. Our study finds that the sign of utility function indeed plays a very important role in studying the relationship between mortality model uncertainty and annuity value.

The rest of this paper is organized as follows. After a recap of the original Yaari's LCM in Section 2, we introduce the proposed extended LCM in Section 3. Section 4 derives the optimal consumption strategy and AEW with discussions about the associated economic implications. A numerical illustration is presented in Section 5. Section 6 concludes the paper. In order to facilitate the reading, all technical proofs are relegated to Appendix A. Throughout, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions, in which $\mathbb{P}$ is a reference probability measure.

2 A recap of Yaari's lifecycle model

Before putting our paper into perspective, this section provides a coarse review of the economic logic behind Yaari's (1965) argument about full annuitization. Consider a rational retiree aged y at time 0. Non-negative RV τ_y denotes the retiree's remaining lifespan, with λ_{y+t} being the corresponding subjective force of mortality at time $t \geq 0$. The survival probability function of τ_y can be computed via

$$ {}_tp_y := \mathbb{P}(\tau_y > t) = \exp\left(-\int_0^t \lambda_{y+s} ds\right). \tag{1} $$

Suppose that the retiree does not have any bequest motive, and neither is willing (or able) to invest in the stock market. As a matter of choice, the retiree can either invest the retirement savings in a bond or an annuity, and then fully consume the payment generated from the holdings. In a complete annuity (CA) market which refers to the availability of a complete set of annuities at actuarially fair prices and any maturities, the rational retiree will convert all the retirement savings into an annuity. An initial retirement saving of $x_0 > 0$ can support a stream of annuity payments described by a function of time $c_A : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$ x_0 = \mathbb{E}\left[\int_0^{\tau_y} e^{-rs} c_A(s) ds\right] = \int_0^\infty e^{-rs}\, {}_sp_y\, c_A(s)\, ds = \int_0^\infty e^{-\int_0^s (r+\lambda_{y+u})du}\, c_A(s)\, ds, \tag{2} $$

where $r > 0$ denotes the instantaneous risk-free interest rate. The time-t actuarial present value of the future annuity payments can be evaluated via

$$ X_A(t) = \mathbb{E}_t\left[\int_t^{\tau_y} e^{-r(s-t)}\, c_A(s)\, ds\right] = \int_t^\infty e^{-\int_t^s (r+\lambda_{y+u})du}\, c_A(s)\, ds, \tag{3} $$

which satisfies the following differential equation:

$$dX_A(t) = \big((r+\lambda_{y+t})X_A(t) - c_A(t)\big)\,dt, \qquad X_A(0)=x_0. \tag{4}$$

Without other means of living, the retiree's wealth trajectory is exactly $\{X_A(t)\}_{t\geq 0}$. Here and thereafter, the subscript "A" attached to the payout function and wealth process is used to emphasize the CA market assumption. (Similarly, we should use subscript "B" to spell out the complete bond market condition that will be introduced in a moment.)

For $\gamma>0$ and $c>0$, let $u(c)=c^{1-\gamma}/(1-\gamma)$ denote the CRRA utility of consumption[1] (when $\gamma=1$, the utility function can be understood as $u(c)=\log c$). The rational retiree will choose an annuity payout for which $c(\cdot)$ maximizes the discounted lifetime utility over consumption:

$$\mathbb{E}\bigg[\int_0^{\tau_y} e^{-\rho s}\,u\big(c(s)\big)\,ds\bigg] = \int_0^\infty e^{-\int_0^s(\rho+\lambda_{y+u})\,du}\times\frac{c(s)^{1-\gamma}}{1-\gamma}\,ds, \tag{5}$$

where $\rho>0$ is the subjective discount rate which may or may not be equal to the risk-free interest rate. The optimal annuity payout function, or equivalently the optimal consumption path, is solved to be

$$c_A^*(t) = x_0\times\frac{\big[\exp\big(-t\,(\rho-r)\big)\big]^{1/\gamma}}{\int_0^\infty\ {}_tp_y\,\big[\exp(-\rho\,s)\,\exp(-r\,s)^{\gamma-1}\big]^{1/\gamma}\,ds}. \tag{6}$$

In analogy to the CA market, it is the complete bond (CB) market wherein pure discount bonds are available for all maturities, but annuities are absent. Thus, the retiree has to rely on bonds as the only means of investment. In this case, we denote the bond payout function by $c_B:\mathbb{R}_+\to\mathbb{R}_+$, which satisfies the budget constraint:

$$x_0 = \int_0^\infty e^{-rs}\,c_B(s)\,ds. \tag{7}$$

The evolution of the corresponding wealth trajectory is given by

$$dX_B(t) = \big(rX_B(t) - c_B(t)\big)\,dt, \qquad X_B(0)=x_0.$$

[1]The LCM considered in Yaari (1965) assumed a more general additive utility, but the choice of CRRA utility in our paper simplifies the presentation.

The rational retiree will base on the same objective function (5) to derive the optimal consumption path which is computed via

$$c_B^*(t) = x_0 \times \frac{\left[{}_t p_y \, \exp\left(-t\,(\rho - r)\right)\right]^{1/\gamma}}{\int_0^\infty \left[{}_t p_y \, \exp(-\rho\, s)\, \exp(-r\, s)^{\gamma-1}\right]^{1/\gamma} ds}. \tag{8}$$

When $\gamma > 1$, then $c_A^*(t)/X_A^*(t) \geq c_B^*(t)/X_B^*(t)$ for all $t > 0$, where X_A^* and X_B^* denote the wealth processes associated with the optimal consumption rules c_A^* and c_B^*, respectively. The above inequality implies that by annuitization, the optimal consumption rate out of the present wealth is higher at all time when the retiree is alive, which leads to the conclusion that the rational retiree should convert all the retirement savings into an annuity upon retirement.

The difference between AEW and x_0 captures the amount of extra initial wealth needed to compensate the absence of annuity in the CB market compared to the CA market. Formally, AEW is defined through

$$V_A\big(x_0\big) = V_B\big(\text{AEW}\big),$$

where for "$\square$" being either "A" or "B",

$$V_\square(x) = \mathbb{E}\left[\int_0^{\tau_y} e^{-\rho s}\, u\big(c_\square^*(s)\big)\, ds\right] = \int_0^\infty e^{-\int_0^s (\rho + \lambda_{y+u})\, du} \times \frac{c_\square^*(s)^{1-\gamma}}{1-\gamma}\, ds, \qquad x_0 = x,$$

denotes respectively the discounted lifetime consumption utility function following the optimal consumption rules (6) and (8), or equivalently, the value functions of the two optimization problems under the CA and CB markets. Under Yaari's LCM, the AEW is given by

$$\text{AEW} = x_0 \times \left[\frac{\int_0^\infty \left[{}_t p_y \, \exp(-\rho\, s)\, \exp(-r\, s)^{\gamma-1}\right]^{1/\gamma} ds}{\int_0^\infty {}_t p_y \left[\exp(-\rho\, s)\, \exp(-r\, s)^{\gamma-1}\right]^{1/\gamma} ds}\right]^{\frac{\gamma}{\gamma-1}} > x_0, \qquad \text{for all } \gamma > 0,$$

which means that the retiree would need a larger amount of initial retirement wealth in the CB market in order to achieve the same level of utility as in the CA market.

3 Formulation of the extended LCM with mortality model uncertainty

As mentioned in the introduction, we aim to advance Yaari's (1965) LCM along two directions, namely recursive utility and mortality model uncertainty. We start off by introducing a continuous-time version of the Epstein-Zin recursive utility (Duffie and Epstein, 1992; Gârleanu and Panageas, 2015) to model the retiree's preferences. For any $t > 0$, define the actuarial subjective discount factor $\alpha_t = \rho + \lambda_{y+t}$, and let $J(t)$ be the discounted future utility at time t. The recursive utility is defined via

$$J(t) = \mathbb{E}_t\bigg[\int_t^\infty f\big(\alpha_s, c(s), J(s)\big)\,ds\bigg], \tag{9}$$

where $c(\cdot)$ denotes the consumption function, $\mathbb{E}_t[\cdot]$ denotes the conditional expectation given the filtration $\mathcal{F}_t$, and

$$f(\alpha, c, v) = \frac{(1-\gamma)\,v}{1-1/\phi}\left[\left(\frac{c}{\big((1-\gamma)v\big)^{\frac{1}{1-\gamma}}}\right)^{1-1/\phi} - \alpha\right] \qquad \text{for } \alpha, c, v > 0, \tag{10}$$

is known as the normalized aggregator of consumption and utility. In Equation (10), $\gamma > 0$ is the relative risk aversion coefficient which measures the retiree's aversion against consumption fluctuations due to the future random state, and $\phi > 0$ is the EIS coefficient which measures the aversion against consumption fluctuations over time in a deterministic world. This spells out the merit of adopting the recursive utility over the CRRA utility in that the retiree's preferences over the timing of the resolution of uncertainty is disentangled from risk aversion, so that we can study the implications of the two risk parameters separately.

It is noted that the recursive utility specified in (9) actually includes the CRRA utility as a special case. This is summarized in the following remark.

Remark 1. *If $\phi = 1/\gamma$, then the aggregator defined in Equation (10) becomes*

$$f(\alpha, c, v) = \frac{c^{1-\gamma}}{1-\gamma} - \alpha v,$$

and the recursive discounted utility (9) reduces to

$$J(t) = \mathbb{E}_t\bigg[\int_t^\infty e^{-\int_t^s \alpha_u du} \times \frac{c(s)^{1-\gamma}}{1-\gamma} ds\bigg], \tag{11}$$

which coincides with the CRRA discounted utility (5) *considered in Section 2. Note that different than the CRRA utility, the recursive utility is not necessarily additive.*

Next we turn to specify a set of plausible probability measures to account for the mortality model uncertainty. Following the mortality perturbation approach proposed in Shen and Su (2019), for any $t > 0$, we introduce an $\mathcal{F}_t$-predictable process $\theta(t) > 0$ to be chosen endogenously by the retiree for adjusting the reference subjective mortality model. Consider an equivalent probability measure $\mathbb{Q}$ which is defined via the Radon–Nikodým derivative:

$$\left.\frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{ \int_0^{t\wedge\tau_y} \big[\log(\theta(s)) - \theta(s) + 1\big]\,\lambda_{y+s}\,ds + \int_0^t \log(\theta(s))\,dZ(s)\right\}, \tag{12}$$

where $Z(s) := \mathbb{1}_{\{\tau_y \le s\}} - \int_0^s \mathbb{1}_{\{\tau_y > u\}}\,\lambda_{y+u}\,du$ is a martingale associated with the single jump process $\mathbb{1}_{\{\tau_y\le s\}}$. By Girsanov's Theorem, from $\mathbb{P}$ to $\mathbb{Q}$, the subjective force of mortality is perturbed from λ_{y+t} to $\lambda^{\mathbb{Q}}_{y+t} = \theta(t)\,\lambda_{y+t}$, $t > 0$. Hence, we refer to $\mathbb{Q}$ as the perturbed measure and $\theta(\cdot)$ as the mortality perturbation function. The survival probability function of τ_y under the perturbed measure $\mathbb{Q}$ can be computed similarly as (1), but with the force of mortality replaced by $\theta(t)\lambda_{y+t}$. That is,

$${}_tp^{\mathbb{Q}}_y := \mathbb{Q}(\tau_y > t) = \exp\left(-\int_0^t \theta(s)\lambda_{y+s}ds\right). \tag{13}$$

The perturbation function $\theta(t)$ controls the difference between the alternative model and the reference model at each time $t > 0$. To quantify the overall discrepancy, the relative entropy is a commonly used statistical measure of distance, suitable for robust control problems. In the context of this current paper, the relative entropy between the perturbed mortality model and the best-estimated reference mortality model can be computed via

$$\begin{aligned}
\mathcal{D}\big(\mathbb{Q}\,|\,\mathbb{P}\big) = \mathbb{E}^{\mathbb{Q}}_t\left[\log\left(\left.\frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\right|_{\mathcal{F}_t}\right)\right] &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^{t\wedge\tau_y} \big[\log(\theta(s)) - \theta(s) + 1\big]\lambda_{y+s}ds + \int_0^t \log(\theta(s))dZ(s)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_0^{t\wedge\tau_y} \big[\theta(s)\log(\theta(s)) - \theta(s) + 1\big]\lambda_{y+s}ds + \int_0^t \log(\theta(s))dZ^{\mathbb{Q}}(s)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_0^{t\wedge\tau_y} \big[\theta(s)\log(\theta(s)) - \theta(s) + 1\big]\lambda_{y+s}ds\right].
\end{aligned}$$

For notational convenience, define

$$g(\theta) := \theta \log \theta - \theta + 1, \qquad \theta > 0,$$

then we can write

$$\mathcal{D}(\mathbb{Q} \,|\, \mathbb{P}) = \mathbb{E}^{\mathbb{Q}}\bigg[\int_0^{t \wedge \tau_y} g\big(\theta(s)\big)\, \lambda_{y+s} ds\bigg]. \tag{14}$$

The $g(\theta)$ function is an infinitesimal generator quantifying the distance between the two probability measures. It is straightforward to check that $g(1) = 0$ and $g'(\theta) = \log \theta$, so if $\theta(t) \equiv 1$, then the perturbed mortality model coincides with the reference mortality model, and so the relative entropy $\mathcal{D} = 0$. For θ_1 and θ_2 both smaller than or both greater than one, if $|\theta_1(t) - 1| > |\theta_2(t) - 1|$ for all $t > 0$, then the perturbed mortality curve associated with θ_1 is farther away from the reference mortality curve than that associated with θ_2, so the corresponding entropy satisfies $\mathcal{D}_1 > \mathcal{D}_2$.

The discussion above is about how to construct a set of alternative mortality models to be considered by the retiree. Another important question is how large the set of alternative mortality models should be. The answer to this question is not unique, but a widely-accepted principle is that alternative models should not be statistically too far away from the reference model which has been best estimated, and those alternative models that are statistically hard-to-be-distinguished from the reference model should be considered more seriously. The penalty approach employed in the literature, such as Maenhout (2004, 2006) and Shen and Su (2019), reflects the aforementioned principle. One can also refer to Lin and Riedel (2021) for an alternative setting of optimal investment and consumption problems with ambiguity, in which no penalty function is involved.

Formally, given the consumption function $c(\cdot)$ and perturbation function $\theta(\cdot)$, we specify a penalty term incurred to the retiree's discounted utility such that

$$J(t) = \mathbb{E}_t^{\mathbb{Q}}\bigg[\int_t^{\infty} f\big(\alpha_s, c(s), J(s)\big)\, ds\bigg] + \frac{1}{\psi} \times \Gamma(t, \theta), \qquad t > 0, \tag{15}$$

where $\psi > 0$ is the robustness preference parameter indicating the extent of the retiree's concern about the mortality model uncertainty, and

$$\Gamma(t,\theta) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\infty} (1-\gamma)\, J(s)\, g\big(\theta(s)\big)\, \lambda_{y+s} ds\right]$$

can be viewed as a scaled counterpart of the relative entropy given in (14). The scaling factor $(1-\gamma)\, J(s)$ is included in the penalty term mainly for an analytic tractability reason. With a fixed ψ, a perturbed model that is farther away from the reference model will cause a larger penalty to the utility, thus the perturbed model is less likely to be accepted by the retiree. Meanwhile, the robustness preference parameter ψ controls the set of alternative mortality models that the retiree is willing to consider. Namely, as ψ increases, the penalty term becomes smaller even for those alternative models that are significantly different from the reference model, so the acceptable set of alternative mortality models expands. In other words, a more ambiguity averse retiree having a high robustness preference ψ, will put less faith on the reference model and consider more diverse alternative mortality models that possess larger relative entropy. When $\psi = 0$, any deviation from the reference model will add an infinitely large penalty to the utility. Hence in this case, the retiree has no ambiguity aversion and fully trusts the reference mortality model. This case corresponds to Yaari's LCM under the recursive utility but without mortality model uncertainty.

Given the mortality perturbed measure and recursive utility, the robust decision problem faced by the retiree can be formulated in terms of the following two contemporaneous courses of action. In one course of action, from the pool of plausible mortality models, the retiree seeks to identify the worst-case mortality perturbation function $\theta^*(\cdot)$ that is most unfavourable to the retiree's consumption utility. In another course of action, the retiree selects the optimal consumption policy to maximize the recursive utility under the worst-case mortality scenario. Collectively, the value function associated with the retirement problem of interest reads as

$$V(t,x) = \max_{c\in\mathfrak{C}}\ \min_{\theta\in\mathfrak{T}} J(c,\theta;\, t,x), \qquad t>0 \text{ and } x>0, \tag{16}$$

where $J(c,\theta;\, t,x)$ is the objective function defined as per (15) with (c,θ) and $x = X(t)$ spelled out to highlight its dependence on the controls and the initial state, $\mathfrak{C}$ and $\mathfrak{T}$ are the admissible spaces for consumption and perturbation functions to be specified in Definition 1 below.

Definition 1. *A consumption function $c(t)$ is said to be admissible if*

- $c(t) \geq 0$*, for all* $t > 0$*;*
- $\int_0^\infty c(s)ds < \infty$*;*
- *the wealth process $X(t)$ associated with $c(t)$ stays positive over the entire planning horizon.*

The space containing all admissible consumption functions is denoted by $\mathfrak{C}$*.*

Moreover, a perturbation function $\theta(t)$ is said to be admissible if

- $\theta(t) > 0$*, for all* $t > 0$*;*
- $\mathbb{Q}$ *is a well-defined probability measure equivalent to* $\mathbb{P}$*.*

The space containing all admissible perturbation functions is denoted by $\mathfrak{T}$*.*

Inspired by the original study of LCM in Yaari (1965), we consider both the CA and CB market conditions, under which the retirement problem (16) satisfies the budget constraints specified in Equations (2) and (7), respectively. In particular, we have a keen interest in studying the following questions:

Q1. If mortality model uncertainty is concerned, what will be the worst-case mortality scenario?

Q2. How mortality model uncertainty will influence the optimal consumption policies?

Q3. What are the implications of mortality model uncertainty on the AEW and the annuity puzzle?

4 Main results

Table 1 summaries the cases that we aim to investigate and compare in this section. Among the four cases, Cases A and B have been studied in Yaari (1965) under the additively separable utility function, but we extend the results to a more general recursive utility framework. The first part of this section studies the optimal consumption policies under Case C and Case D, thus answering Questions Q1 and Q2. The comparisons between Case A and Case C, and between Case B and Case D, are considered in the second part of this section, which answers Question Q3.

4.1 Optimal consumption policies under mortality model uncertainty

In this section, by applying the dynamic programming principle to the max-min problem in Equation (16), we solve the optimal consumption strategies for the retiree described in Section 3. The succeeding theorem

	Ambiguity neutral	Ambiguity averse
Complete annuity market	Case A	Case C
Complete bond market	Case B	Case D

Table 1: Summary of the four different retirement cases considered in this current paper.

summarizes the main mathematical results. Recall that quantities related to the CA and CB market conditions are distinguished by the subscripts "A" and "B", respectively. The robustness preference parameter ψ is specified in the optimal decision rules in order to highlight their dependencies with the retiree's aversion of mortality model uncertainty which is the major objective in our paper.

Theorem 1. *Suppose the retirement environment as per the description in Section 3, the worst-case perturbation function associated with the optimization problem (16) can be computed via*

$$\theta^*_A(t\,;\psi) = \theta^*_B(t\,;\psi) \equiv \theta^*(\psi) = \exp\big(\frac{\psi}{1-1/\phi}\big) \qquad \textit{for all } t > 0,$$

where $\psi > 0$ is the robust preference parameter and $\phi > 0$ is the EIS coefficient. The optimal robust consumption strategies are given by

$$c^*_\square(t;\psi) = c^*_\square(0;\psi) \times \exp\bigg\{\int_0^t \big[(1 - G_\square(\psi))\lambda_{y+u} - (\rho - r)\phi\big]\,du\bigg\}, \tag{17}$$

where "$\square$" can be either "A" or "B", and

$$G_A(\psi) = (1-\phi) + G_B(\psi), \qquad G_B(\psi) = \phi\,\theta^* + \frac{1-\phi}{\psi}\,g(\theta^*).$$

Moreover, the optimal initial consumption quantities can evaluated via

$$c^*_\square(0\,;\psi) = \frac{x_0}{K_\square(\psi)}, \qquad \textit{with } K_\square(\psi) = \int_0^\infty \exp\bigg(-\int_0^s (\beta + G_\square(\psi)\,\lambda_{y+u})du\bigg)\,ds, \tag{18}$$

in which $\beta = (1-\phi)\,r + \phi\,\rho$ is the weighted average between the risk-free interest rate and subjective discount rate.

Given the optimal robust strategies $c^*_\square$ and θ^*, the value function (16) at the present time can be *computed via*

$$V_{\square}(0, x_0\,; \psi) = \Big[K_{\square}(\psi)\Big]^{-\frac{1-\gamma}{1-\phi}} \frac{x_0^{1-\gamma}}{1-\gamma},$$

where "$\square$" is either "A" or "B".

Proof. See Appendix A. $\square$

A careful inspection of the optimal strategies outlined in Theorem 1 reveals the following findings. First, the worst-case perturbation functions are identical under the CA and CB market conditions. This is because the optimization problems under the two market conditions differ only up to the associated budget constraints which depend on the risk-free interest rate and the reference mortality curve, while the worst-case perturbation functions $\theta_A^*(\cdot)$ and $\theta_B^*(\cdot)$ only depend on the EIS coefficient ϕ and robustness preference parameter ψ but not the other parameters. Interestingly, the worst-case perturbation functions are constant over time, and the optimal perturbed mortality model is given by

$$\lambda_{y+t}^* = \theta^* \times \lambda_{y+t}, \qquad t > 0, \tag{19}$$

corresponding to a proportional shift of the best-estimated reference mortality curve. In the language of actuarial mathematics, transformation (19) is also known as the proportional hazard distortion (Wang, 1996). The proportional transform is commonly used to test the sensitivity of mortality assumption in retirement research (see, e.g., Han and Hung, 2021; Milevsky and Young, 2007; Shen and Su, 2019). This type of transform is applied in earlier studies mainly because of its inherent simplicity. However, our finding herein provides a theoretical justification for the choice of proportional shock in sensitivity tests, which is perhaps sufficiently conservative for covering the worst-case mortality scenario, so far at least when Yaari's LCM is concerned.

Second, the selection of the worst-case mortality shock θ^* depends on the interplay between the EIS coefficient ϕ and the robustness preference ψ. Recall that ψ^{-1} determines the amount of faith that a retiree puts on the reference model. The larger the value of ψ is, the larger the value of $|\theta^* - 1|$ becomes, implying that a retiree concerning more about model uncertainty will rationally consider the worst-case perturbed model to be farther away from the reference model. Meanwhile, if $(1 - 1/\phi) < 0$, or equivalently $\phi < 1$, then $\theta^* < 1$, meaning that the worst-case perturbed probability measure corresponds to an improved mortality scenario, and vice versa. Empirical study has already suggested that the EIS coefficient ϕ for an investor is typically less than 1 (e.g., Yogo, 2004). In this sense, the longevity risk is more relevant to the context of this current paper.

What is more, as mentioned in Remark 1, if $1/\phi = \gamma$, then the recursive utility reduces to the CRRA utility. In this case, the worst-case mortality shock becomes

$$\theta^*(t) = e^{\frac{\psi}{1-\gamma}}, \qquad t > 0. \tag{20}$$

Based on a more complicated LCM, Shen and Su (2019) adopted the same penalty approach to obtain the optimal robust consumption-investment-insurance strategies when there are both economics and mortality models uncertainty, and the investor's preference is depicted by the CRRA utility. Though analytical expression is not available for $\theta^*(t)$ in their case due to the involved mathematical complexity, it is shown in Shen and Su (2019) that $\theta^*(t) < 1$ if the risk aversion parameter of the CRRA utility satisfies $\gamma > 1$, and vice versa. In this sense, the implication of (20) is consistent with the finding in Shen and Su (2019). Adopting a more general recursive utility framework in this current paper allows us to distinguish the EIS coefficient from the risk aversion parameter. As a result, we further clarify the assertion in Shen and Su (2019) by theorizing that, under Yaari's LCM, whether the worst-case perturbed mortality curve corresponds to a mortality risk scenario (i.e., $\theta^* > 1$) or a longevity risk scenario (i.e., $\theta^* < 1$) depends solely on the EIS parameter, but not the risk aversion parameter. Our discussion thus far in this current section answers Question Q1 posted at the end of Section 3.

Next, we focus on the optimal consumption strategies obtained in Theorem 1. Curiously, the optimal consumption function (17) is independent of the risk aversion parameter γ of the recursive utility. To see the reason, recall that γ captures the risk aversion against consumption fluctuations due to the uncertain state in the future. As the retiree has converted all retirement savings into either an annuity or a bond investment at the beginning of the planning horizon, there is no uncertainty involved in the future cash flows as long as the retiree is alive. Meanwhile, the stochastic lifetime acts as an extra deterministic discount factor in the expected utility optimization problem (see, Equation (3)). Consequently, the optimal consumption strategies $c_A^*(\cdot)$ and $c_B^*(\cdot)$ depend only on the EIS coefficient together with other actuarial parameters including mortality and discount rates, but not the risk aversion parameter γ.

It is also interesting to study the patterns of the optimal consumption pathways over time. The succeeding corollary summarizes the monotonicity property for the optimal consumption function (17).

Corollary 2. *For* $\square \in \{A, B\}$ *and any* $t \geq 0$*, if* $(1-G_\square(\psi))\lambda_{y+t} \geq (\rho-r)\phi$*, then the optimal consumption function* $c^*_\square(t;\psi)$ *is increasing in* t*, and vice versa.*

Proof. The proof follows immediately from the expression of optimal consumption function (17). $\square$

The following lemma is of auxiliary importance in our later discussion.

Lemma 3. *For all* $\psi > 0$*, the functions* $G_A(\psi)$ *and* $G_B(\psi)$ *are decreasing in* ψ *if the EIS* $\phi < 1$*, and increasing in* ψ *otherwise. Further, if* $\phi < 1$*, then*

$$1 - \phi \leq G_A(\psi) \leq 1 \qquad \text{and} \qquad 0 \leq G_B(\psi) \leq \phi.$$

Otherwise,

$$1 \leq G_A(\psi) \leq \infty \qquad \text{and} \qquad \phi \leq G_B(\psi) \leq \infty.$$

Proof. See Appendix A. $\square$

Corollary 2 and Lemma 3 together imply that the optimal consumption paths may present an asymmetric U-shaped pattern over time. For instance, if $\phi < 1$ and $\rho > r$, then at the beginning of the retirement phase when the force of mortality is still low, the condition of Corollary 2, $(1-G_\square(\psi))\lambda_{y+t} \leq \phi\,(\rho-r)$, is satisfied, thus the optimal consumption function may be decreasing in time. But the optimal consumption will ultimately become increasing in time as the mortality rate grows during the later stage of retirement.

The next assertion compares the optimal consumption rules between the CA market and the CB market (i.e., Case C versus Case D in Table 1). Let $\pi(t) = c(t)/X(t)$ be the consumption-to-wealth ratio at time $t > 0$, which indicates the retiree's propensity to consumption out of the present wealth. The consumption-to-wealth ratios associated with the optimal robust strategies reported in Theorem 1 is denoted by $\pi^*_\square(t\,;\psi) = c^*_\square(t\,;\psi)/X^*_\square(t\,;\psi)$, where $X^*_\square(\cdot\,;\psi)$ is the corresponding wealth process, $\square \in \{A, B\}$.

Proposition 4. *The following relationships hold for the optimal strategies outlined in Theorem 1:*

I. If the EIS $\phi \begin{Bmatrix} < \\ > \end{Bmatrix} 1$*, then* $\pi^*_A(t\,;\psi) \begin{Bmatrix} > \\ < \end{Bmatrix} \pi^*_B(t\,;\psi)$ *for all* $t > 0$ *and* $\psi > 0$*.*

II. If the EIS $\phi < 1$ *(resp.* $\phi > 1$*), then there exists a time epoch* $t^* > 0$ *such that* $c^*_A(t\,;\psi)$ *is greater (resp. smaller) than or equal to* $c^*_B(t\,;\psi)$ *for* $t \le t^*$*, but the inequality is reversed for* $t > t^*$*.*

III. For any EIS $\phi > 0$ *and robustness preference* $\psi > 0$*,* $V^*_A(0, x_0\,;\psi) > V^*_B(0, x_0\,;\psi)$*.*

Proof. See Appendix A. □

Proposition 4 answers Question Q2 posted in Section 3. Specifically, it shows that even with mortality model uncertainty, annuitization may increase the optimal consumption rate at all times if the EIS parameter $\phi < 1$ which is a realistic case supported by Yogo (2004). Although there is a twisted pattern in the comparison of the absolute consumption amounts between the CA and CB markets, the discounted lifetime utility in consumption is always higher by annuitization.

In concluding this subsection, we report another important quantity in our study, namely the AEW in the state of mortality model uncertainty and recursive utility.

Theorem 5. *Suppose the retirement environment as per the description in Section 3. Given the initial wealth* $x_0 > 0$*, the annuity equivalent wealth can be computed via*

$$\mathrm{AEW}(\psi) = x_0 \left[\frac{K_B(\psi)}{K_A(\psi)}\right]^{1/(1-\phi)}, \qquad \psi > 0.$$

Proof. See Appendix A. □

As mentioned earlier, the full annutization (resp. bond investment) in the CA (resp. CB) market removes the uncertainty in the future cash flow, so the AEW does not depend on the risk aversion parameter γ.

Remark 2. *Note that the AEW reported in Theorem 5 is always greater than or equal to* x_0*. To see why, recall that* $G_A(\cdot) = (1-\phi) + G_B(\cdot)$*, which implies* $G_A(\cdot) \ge G_B(\cdot)$ *(resp.* $G_A(\cdot) > G_B(\cdot)$*) if* $\phi \le 1$ *(resp.* $\phi > 1$*). Thus, it always holds* $\left[K_A(\cdot)\right]^{1/(1-\phi)} \le \left[K_B(\cdot)\right]^{1/(1-\phi)}$ *according to Equation (18). Speaking plainly, the inequality means that annuitization will induce an incremental utility to the retiree even when there is a presence of mortality ambiguity aversion.*

4.2 Implications of mortality model uncertainty

The prior section is devoted to the study of the optimal robust consumption strategies for Yaari's LCM equipped with the recursive utility. In this section, we proceed to study the implications of mortality model uncertainty on the optimal consumption rules as well as the AEW.

Proposition 6. *For any fixed $t > 0$ and $\square \in \{A, B\}$, the optimal consumption-to-wealth ratio $\pi^*_\square(t\,;\psi)$ is decreasing in the robustness preference parameter ψ if the EIS parameter $\phi < 1$, or increasing otherwise.*

Proof. See Appendix A. $\square$

The above result shows that when the EIS $\phi < 1$, increasing mortality ambiguity aversion will lower the percentage of consumption out of the present wealth at every instant, no matter whether or not annuity is purchased. This is because if $\phi < 1$, then the worst-case perturbation function $\theta^* < 1$, which corresponds to a longevity risk scenario. Consequently, the retiree reduces the consumption rate in order to lower the risk of outliving the retirement savings. Alternatively, if the EIS $\phi > 1$, then the worst-case perturbed mortality curve corresponds to a mortality risk scenario. In order to maximize the discounted lifespan utility, the rational retiree will choose to increase the consumption ratio.

In addition to the relative consumption ratio, we are also interested in studying the impacts of mortality model uncertainty on the absolute consumption amounts. Intuitively, if the robustness preference parameter ψ changes, the optimal robust consumption rules should decrease for some t while increase for the others, so that the budget constraint (2) or (7) can be maintained. The next assertion shows that in the case of $\phi < 1$ where longevity risk is more concerned, the retiree will rationally reduce the consumption amount at the beginning of the retirement phase so as to keep more savings for the future. In another case where the EIS parameter $\phi > 1$ and thus mortality risk is concerned, then the rational retiree will choose to increase the consumption amount at the beginning of the retirement phase so as to make sure that a desirable level of consumption utility can be gained before death.

Proposition 7. *For $\square \in \{A, B\}$, if the EIS $\phi < 1$, then there exists a time epoch $t^* > 0$, the optimal consumption function $c^*_\square(t\,;\psi)$ reported in Theorem 1 is decreasing in ψ for all $t \leq t^*$ and becomes increasing in ψ for $t > t^*$. Otherwise, the behavior of the optimal consumption function is reversed.*

Proof. See Appendix A. $\square$

Remark 3. *Consider the case where $\psi \to 0$ meaning that the retiree has no mortality ambiguity aversion, then*

$$\lim_{\psi\to 0} G_A(\psi) = 1 \qquad \text{and} \qquad \lim_{\psi\to 0} G_B(\psi) = \phi.$$

The optimal consumption functions become

$$\lim_{\psi\to 0} c_A^*(t\,;\psi) = x_0 \times \frac{\exp\big(-\int_0^t \phi\,(\rho - r)\,du\big)}{\int_0^\infty \exp\big(-\int_0^s (\beta + \lambda_{y+u})\,du\big)\,ds} \tag{21}$$

and

$$\lim_{\psi\to 0} c_B^*(t\,;\psi) = x_0 \times \frac{\exp\big\{-\int_0^t [\phi\,(\rho - r + \lambda_{y+u}) - \lambda_{y+u}]\,du\big\}}{\int_0^\infty \exp\big(-\int_0^s (\beta + \phi\,\lambda_{y+u})\,du\big)\,ds}. \tag{22}$$

Further, suppose that the EIS and risk aversion parameters satisfy $\phi = 1/\gamma$ so the recursive utility reduces to the CRRA utility, then Equations (21) and (22) collapse respectively to the optimal consumption functions (6) and (8) under the classical Yaari's LCM.

Finally, the impacts of mortality model uncertainty on the value functions are considered. Recall again that if the EIS parameter $\phi < 1$, then the associated worst-case perturbed mortality model corresponds to a longevity scenario, and as ψ increases, the concern about longevity risk becomes stronger, so annuity should become more valuable. In other words, the AEW increases with the robustness preference parameter ψ when the EIS parameter $\phi < 1$, and vice versa. The succeeding assertion confirms our conjecture.

Proposition 8. *If $\phi < 1$ (resp. $\phi > 1$), then* $\mathrm{AEW}(\psi)$ *is increasing (resp. decreasing) in $\psi > 0$.*

Proof. See Appendix A. $\square$

Based on Proposition 8, we argue that mortality model uncertainty is a potential contributor to the enduring puzzle of low annuity demand. Namely, if the uncertainty surrounding the mortality model—even though it has been best estimated—is overlooked by the retiree, then the value of annuity may be understated when $\psi < 1$. Our study acknowledges that retirees should not place full conviction on a specific mortality assumption. Otherwise, the longevity risk inherent in retirement planning will be underestimated. Educating investors to recognize the mortality model uncertainty may be one of the possible ways to resolve the issue of low annuity demand in the present retirement market. The study in this current subsection answers Question Q3 posted in Section 3.

4.3 Other considertations

It will be also interesting to investigate the dependence of AEW on other parameters such as the subjective discount rate and interest rate. With a slight abuse of notation, in what follows, we use $\mathrm{AEW}(\rho)$ and $\mathrm{AEW}(r)$ to denote the AEW as a function of ρ or r.

Proposition 9. *(i) If $\phi < 1$ (resp. $\phi > 1$), then $\mathrm{AEW}(\rho)$ is decreasing (resp. increasing) in $\rho > 0$.*

(ii) For any $\phi > 0$, $\mathrm{AEW}(r)$ is decreasing in $r > 0$.

Proof. See Appendix A. □

A comparison of Assertions (i) and (ii) shows that the impact of the subjective discount rate ρ on the AEW is interacted with the EIS, while that of the risk-free interest rate r is independent of the EIS. This difference is determined by the subjective and objective natures of the two parameters ρ and r. The former has a natural interaction with the EIS, which is also a subjective parameter, reflecting the retiree's aversion against consumption flucations.

Moreover, we study how the AEW responds to changes in the mortality curve $\{\lambda_{y+t}\}_{t\geq 0}$. In general, it is very challenging to develop a general result for ordering the AEW associated with two arbitrary mortality curves. Thereby, we confine ourselves to a simpler case in which the benchmark mortality curve is shocked by a parallel shift $\{\lambda_{y+t} + \Delta\}_{t\geq 0}$, $\Delta > 0$. The succeeding proposition studies the sensitivity of $\mathrm{AEW}(\Delta)$ as a function of the magnitude of the shock. The proposition contains an assumption about increasing mortality rate over time, which is very natural for modeling the future lifetime of a retiree.

Proposition 10. *For any $\phi > 0$ and suppose that λ_{y+t} is increasing in t, then $\mathrm{AEW}(\Delta)$ is increasing in $\Delta \geq 0$.*

Proof. See Appendix A. □

The above result hints that a worsen mortality pathway may yield an increase in the AEW. An intuitive explanation behind the result is that a worsen mortality pathway lowers the price of an annuity and thus increases the utility increment via annuitization.

5 Numerical illustration

In this section, we illustrate numerically the theoretical findings established thus far. Suppose that the rational retiree of interest is now aged 65 and endowed with a retirement saving of \$100 (thousand). We estimate the baseline mortality model using the celebrated Gompertz law (Gompertz, 1825; also see Milevsky, 2020 for a recent development):

$$\lambda_x^{\mathrm{GM}} = w_1 \, \exp(w_2 \, x), \qquad x, w_1, w_2 > 0. \tag{23}$$

We fit the Gompertz mortality model (23) into the 2015–2019 U.S. mortality table extracted from the Human Mortality Database[2]. The parameters are estimated to be $\widehat{w}_1 = 5.01 \times 10^{-5}$ and $\widehat{w}_2 = 8.39 \times 10^{-2}$ for female, and $\widehat{w}_1 = 8.10 \times 10^{-5}$ and $\widehat{w}_2 = 8.25 \times 10^{-2}$ for male. Figure 1 depicts the probability density function as well as the survival probability associated with the fitted mortality model. As shown in the right panel of Figure 1, the female survival probability function dominates that of male, implying that female is more likely to survive longer than male at any future time points.

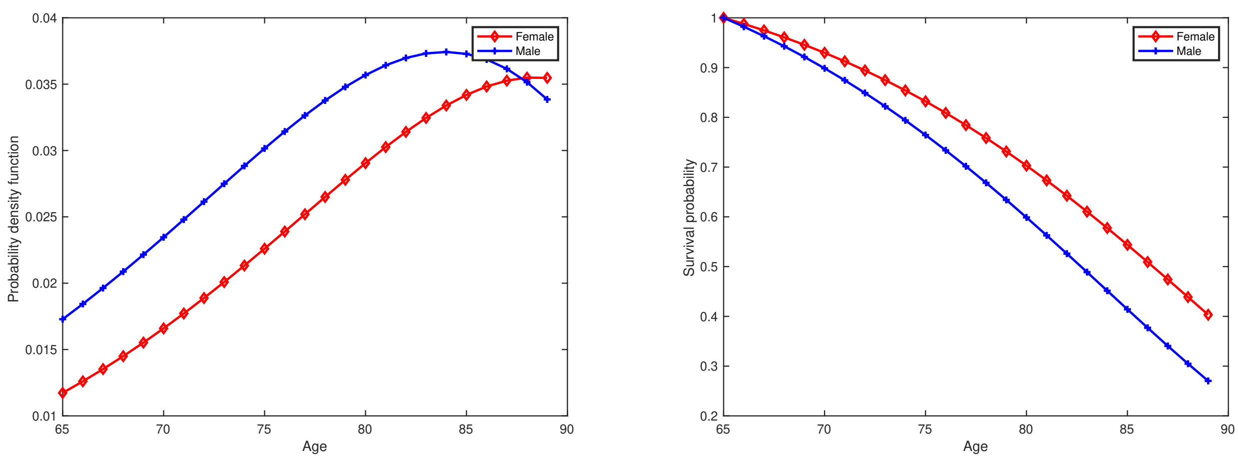

Figure 1: The probability density functions (left panel) and survival probability functions (right panel) for the female and male retirees' remaining life time RV's.

We set the risk-free rate to be 1.9% according to the U.S. cash rate in the 2021 Long-Term Capital Market Assumptions report published by the J.P. Morgan Asset Management. We assume the subjective discount rate to be 3% which is a standard choice in the related literature. Indeed, Frederick et al. (2002) conducted a comprehensive literature review on estimated discount rates and found a predominance of

[2]Source: http://www.mortality.org/

high discount rates, being well above market interest rate. The choices of the EIS coefficient ϕ and robustness parameter ψ are rather subjective, mainly depending on the retiree's individual risk profile. Motivated by the empirical study in Yogo (2004), we set $\phi = 0.5$ and $\psi = 1$ as the baseline parameters and then use sensitivity tests to study their implications on the optimal consumption strategies and the AEW.

Based on the aforementioned setting, Figure 2 presents the optimal consumption pathways computed according to Theorem 1. We find that the optimal consumption amount and consumption-to-wealth ratio for male are higher than that of female, under both the CA and CB market conditions. This is intuitive because the life expectancy of female is longer than male, the female retiree will rationally lower the consumption in order to save more wealth for future. Meanwhile, for both the female and male retirees, the optimal consumption amount under the CA market is higher than that under the CB market at the beginning of the retirement phase, but becomes lower after certain time point. This occurs because by purchasing an annuity, the retiree earns higher returns than by purchasing a bond due to the mortality credit. However, when it comes to the optimal consumption-to-wealth ratio, it always holds that $\pi_A^* > \pi_B^*$. These observations coincide with the theoretical investigation in Proposition 4.

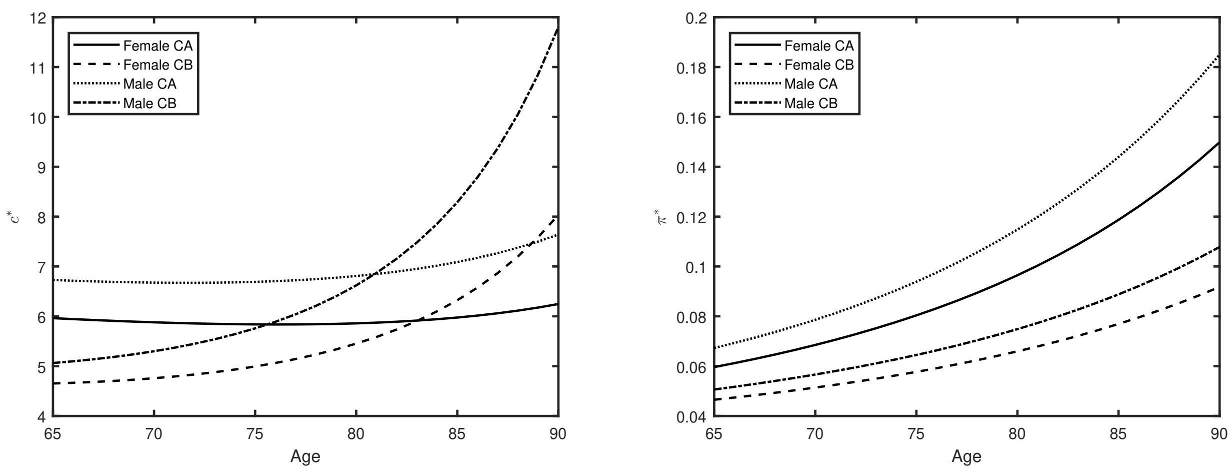

Figure 2: The optimal consumption amounts (left panel) and optimal consumption-to-wealth ratios (right panel) under the baseline parameters.

Next we are going to study the sensitivities of the optimal consumption policies in response to varying EIS parameter $\phi \in (0, 1)$. As mentioned in Section 1.1, an empirical study conducted by Yogo (2004) suggested that the EIS parameter for a typical investor falls into this range. We have already seen in Figure 2 that the optimal consumption rules between female and male have very similar patterns, thus we should only focus on the female retiree herein. Figure 3 displays the optimal consumption rules with different values of the EIS parameter. We observe that under the CA market condition, a smaller value of EIS leads to a flatter optimal consumption function over time, and the optimal consumption-to-wealth ratio tends to be higher. However, the optimal consumption pathways under the CB market do not seem to have a monotonic pattern in response to the changing EIS coefficient. This is caused by the fact that the EIS ϕ determines not only the optimal consumption decision but also the worst-case perturbed mortality scenario, which complicates the impacts of ϕ on the optimal robust consumption rules.

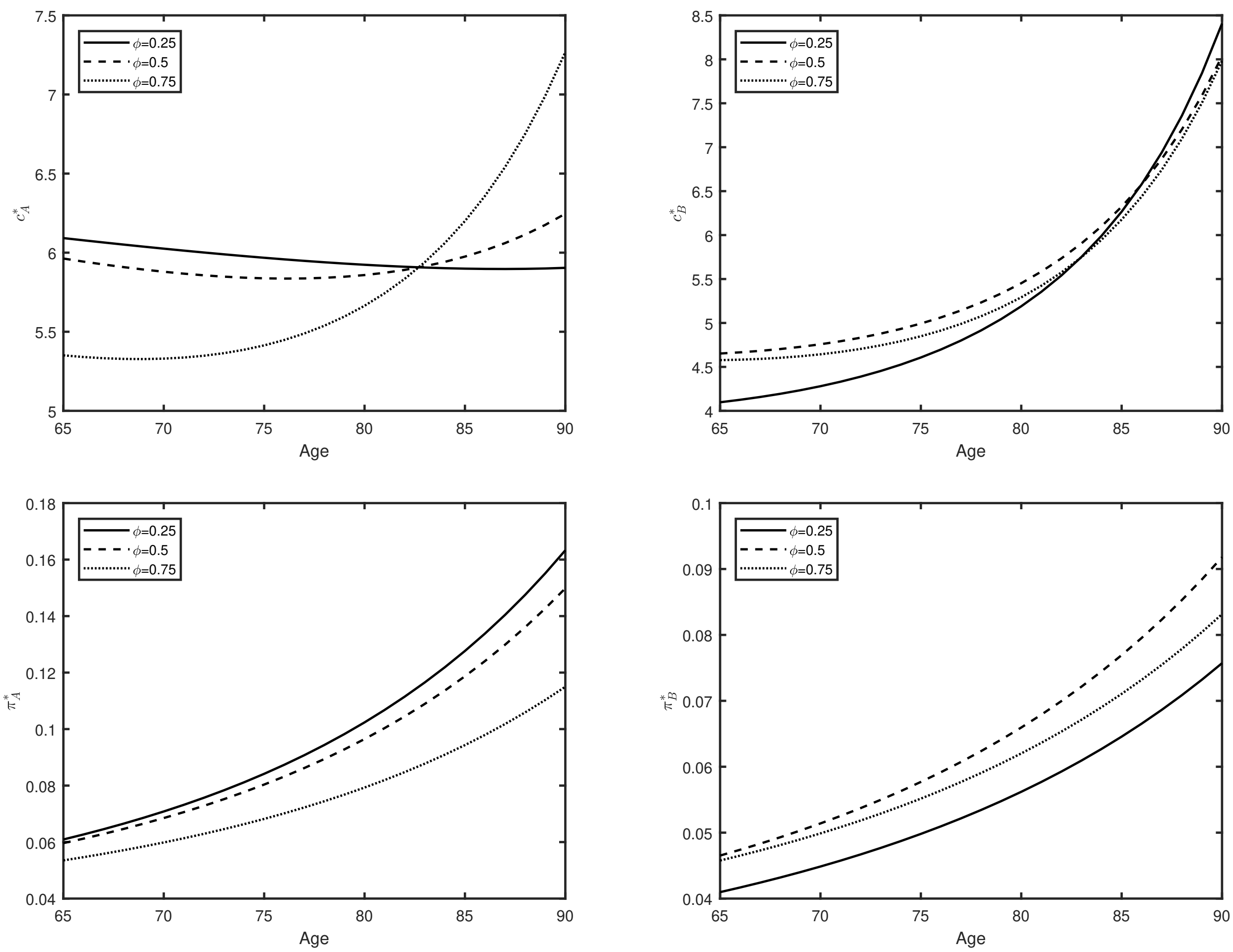

Figure 3: The optimal consumption amounts (first row) and optimal consumption-to-wealth ratios (second row) with varying robustness preference parameter $\phi \in \{0.25, 0.5, 0.75\}$.

Different than the EIS coefficient, the impact of the robustness preference parameter $\psi > 0$ is much more predictable. According to Figure 4, we find that a lower robustness preference ψ increases the optimal consumption amount only at the beginning of the retirement phase, but increases the optimal consumption-to-wealth ratio at all times. This is consistent with the theoretical results in Propositions 6 and 7. We refer the readers to Section 4.2 for intuitive explanations of the observed patterns.

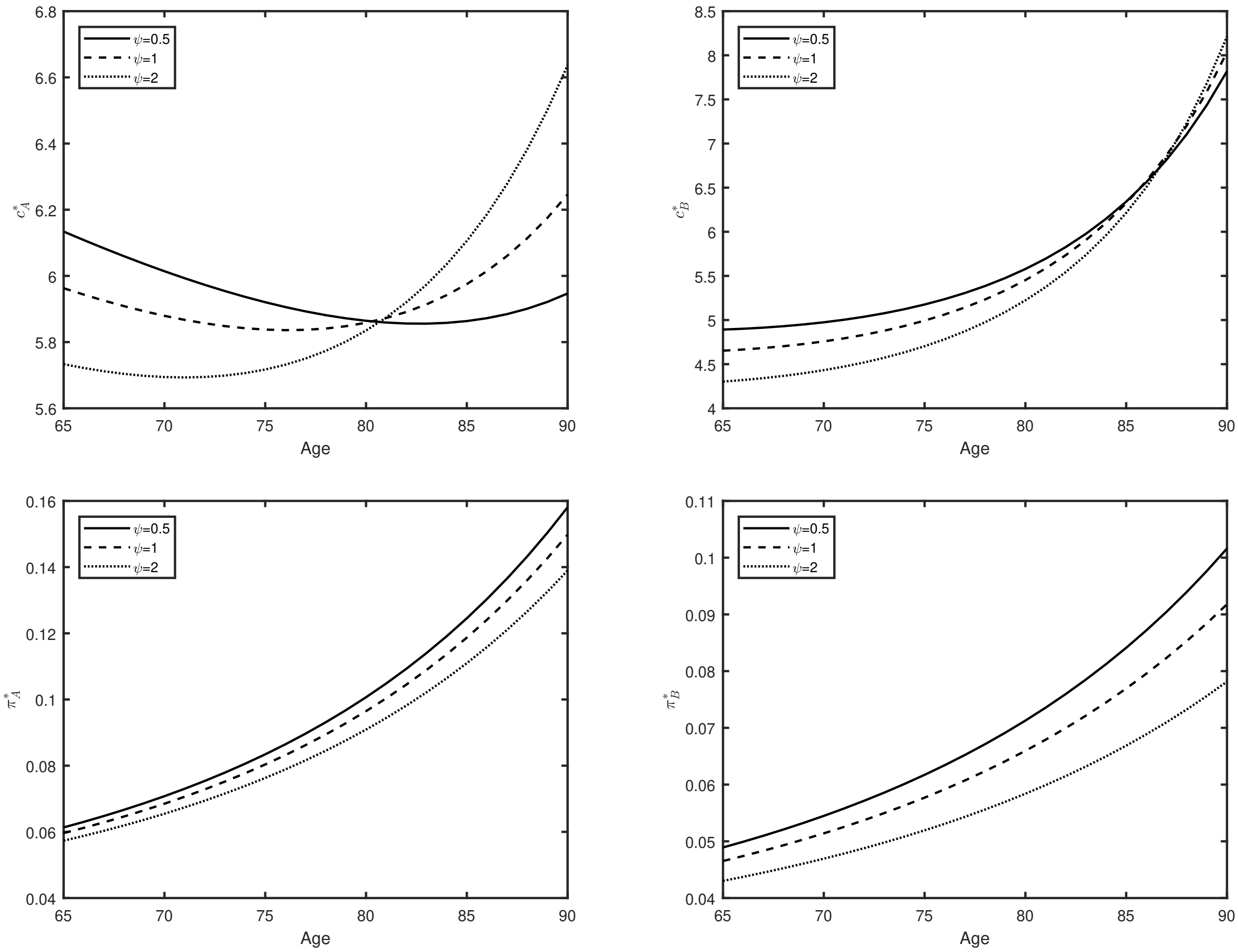

Figure 4: The optimal consumption amounts (first row) and optimal consumption-to-wealth ratios (second row) with varying EIS coefficient $\psi \in \{0.5,\, 1,\, 2\}$.

Finally, we study the impact of the robustness preference parameter ψ on the AEW. Two cases are considered. In the first case, we stick with the baseline EIS $\phi = 0.5$ which is smaller than one and indicates a retiree's preference for more stable consumption pattern over time, then the AEW is increasing in the retiree's robustness preference parameter ψ. In another case where $\phi = 1.5$ which is greater than one, indicating that a retiree has a higher tolerance for future consumption fluctuations, then the relationship between ψ and AEW is reversed. In other words, the aversion of future consumption fluctuations caused by mortality model misspecification can be translated into a fear of longevity risk when $\phi < 1$. A growing concern about mortality model uncertainty will essentially cause the AEW to increase. The observed pattern coincides with our assertion in Proposition 8 (also see more detailed discussions about the economic implications at the end of Section 4.2). It is also worth mentioning that in Figure 5, the magnitude of the AEW is as high as 190% of the intial wealth, which is consistent with numerous results reported in the literature (see e.g., Brown, 2001; Milevsky and Huang, 2018). It is caused by the fact that LCM is based on a utility maximization framework, and utility functions are non-linear and concave. Thus, an immoderate amount of wealth is needed for compensating the utility reduction due to the absence of annuity within the CB market. Moreover, we note that the AEW for the female retiree is always lower than that for the male retiree. The reason is that the female retiree has lower mortality rate than the male retiree, so the annuity price for the female retiree is higher, which lowers the utility gained by annuitization. This observed pattern was hinted by Proposition 10.

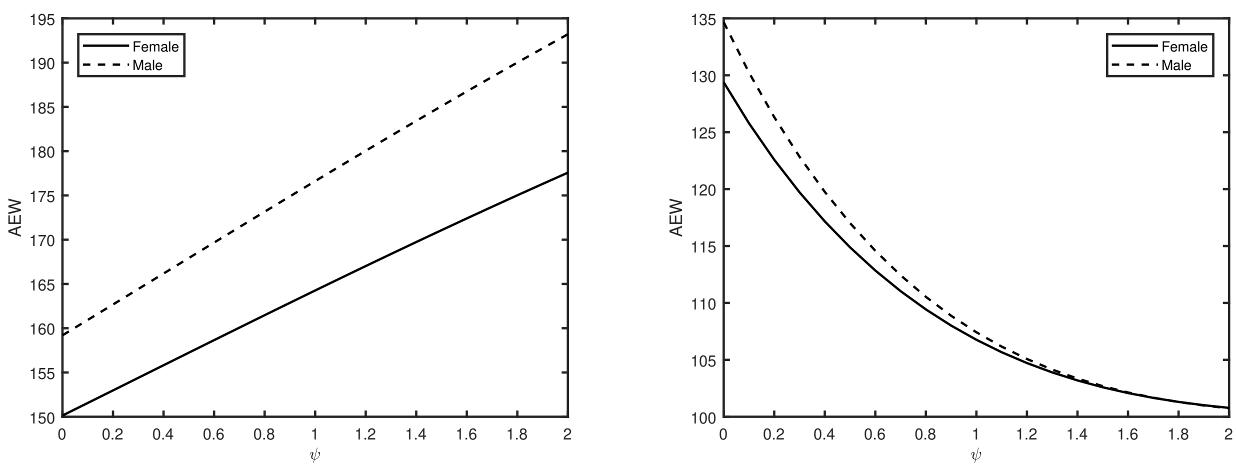

Figure 5: The AEW's for the female and male retirees when the EIS $\phi = 0.5$ (left panel) and $\phi = 1.5$ (right panel).

6 Conclusions

In this paper, we proposed and studied a revamped LCM in which there is an incorporation with recursive utility and mortality model uncertainty. We calculated the optimal robust consumption rule as well as the associated AEW in explicit forms. Our major economic findings include the following. First, we found that for a typical retiree having EIS in consumption smaller than one, the worst-case perturbed mortality curve corresponds to an improved mortality scenario, meaning that the longevity risk is more of a concern than mortality risk in the presence of mortality model uncertainty. Second, under mortality model uncertainty, annuity should be still attractive to retirees in the sense that by annuitization, the optimal consumption rate becomes higher. However, mortality ambiguity aversion will lower the optimal consumption rate. Third, if mortality model uncertainty is disregarded by retirees, then the value of annuity will be understated, potentially causing a lower than expected annuity demand.

There are several topics for future research. First, it will be very interesting to extend the current study from a deterministic mortality framework to a stochastic one. It is likely that the optimal consumption and annuity demand would depend on a retiree's ambiguity and risk aversion levels. How the two different types of aversion behaviors interact with each other is a worthwhile topic for further study. Second, while the current paper focuses on life annuity, it will be interesting to investigate the retiree's optimal decisions when facing multiple retirement income products, such as deferred annuity, variable annuity, and tontine. In particular, it is found in recent literature (see, e.g., Chen et al., 2019, 2021, 2020) that compared with life annuity, tontine may be a more attractive option for retirees to drawdown their retirement savings. Whether or not this result remains valid in presence of endogenous mortality uncertainty is an important follow-up research question. Third, in later life, retirees have to cope with not only mortality risk but also health risk (e.g. long-term care risk). It will be interesting to incorporate a health state transition model into the analysis and study the joint impacts of mortality and health models uncertainty on the optimal consumption rule and optimal demand for annuity and long-term care insurance.

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Appendix A Proofs

Proof of Theorem 1. We only provide the proof for the CA market. The proof for the CB market is essentially the same, thereby omitted. To simplify our notation, we suppress the subscript "A" in functions V and K in the sequel.

Given $X_t = x > 0$, the Hamilton–Jacobi–Bellman (HJB) equation for the optimization problem (16) can be specified as

$$\max_{c\in\mathfrak{C}}\min_{\theta\in\mathfrak{T}}\left\{V_t + V_x\big[(r+\lambda)x - c\big] + \frac{(1-\gamma)V}{1-1/\phi}\left[\left(\frac{c}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{1-1/\phi} - (\rho+\theta\lambda)\right] + \frac{1-\gamma}{\psi}g(\theta)\lambda V\right\} = 0.$$

By the first order conditions of c and θ, we get

$$-V_x + \frac{(1-\gamma)V}{1-1/\phi}\frac{1-1/\phi}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\left(\frac{c}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{-1/\phi} = 0 \tag{24}$$

and

$$-\lambda\frac{(1-\gamma)V}{1-1/\phi}+\frac{1-\gamma}{\psi}g'(\theta)\lambda V=0. \tag{25}$$

We conjecture the following *ansatz*

$$V(t,x\,;\psi)=\Big[K(t\,;\psi)\Big]^{-\frac{1-\gamma}{1-\phi}}\,\frac{x^{1-\gamma}}{1-\gamma}\qquad\text{with } K(\infty\,;\psi)=0,$$

is a solution to the HJB equation above. Then from Equations (24) and (25), we obtain

$$c^*(t\,;\psi)=\frac{x}{K(t\,;\psi)}\qquad\text{and}\qquad\theta^*(t\,;\psi)\equiv\theta^*(\psi)=e^{\frac{\psi}{1-1/\phi}}. \tag{26}$$

To solve K, we substitute c^* and θ^* back to the HJB equation and get

$$-\frac{1}{1-\phi}K_t(t\,;\psi)\big[K(t\,;\psi)\big]^{-\frac{1-\gamma}{1-\phi}-1}x^{1-\gamma}+\big[K(t\,;\psi)\big]^{-\frac{1-\gamma}{1-\phi}}x^{1-\gamma}\left[(r+\lambda)-\frac{1}{K(t\,;\psi)}\right]$$
$$+\frac{\big[K(t\,;\psi)\big]^{-\frac{1-\gamma}{1-\phi}}x^{1-\gamma}}{1-1/\phi}\left[\frac{1}{K(t\,;\psi)}-\big(\rho+\theta^*(\psi)\lambda\big)\right]+\frac{\lambda}{\psi}g\big(\theta^*(\psi)\big)[K(t\,;\psi)]^{-\frac{1-\gamma}{1-\phi}}x^{1-\gamma}=0.$$

Standard algebraic manipulation yields

$$K_t(t\,;\psi)-(1-\phi)\left[(r+\lambda)-\frac{\rho+\theta^*(\psi)\,\lambda}{1-1/\phi}+\frac{\lambda}{\psi}g(\theta^*(\psi))\right]K(t\,;\psi)+1=0,$$

whose solution is given by

$$K(t\,;\psi)=\int_t^\infty\exp\left(-\int_t^s(\beta+G(\psi)\,\lambda_{y+u})du\right)ds.$$

Let $X^*(\cdot)$ be the wealth trajectory associated with the optimal consumption function derived above. The dynamic of the optimal consumption c^* evolves as

$$dc^*(t;\psi)=\frac{1}{K(t;\psi)}\big[-(\beta+G(\psi)\lambda_{y+t})+(r+\lambda_{y+t})\big]\,X^*(t;\psi)dt$$
$$=\big[(1-G(\psi))\,\lambda_{y+t}-\phi(\rho-r)\big]\,c^*(t;\psi)dt.$$

Hence,

$$c^*(t;\psi) = c^*(0;\psi)\,\exp\left\{\int_0^t \big[(1-G(\psi))\lambda_{y+u} - (\rho-r)\phi\big]\,du\right\}.$$

The proof is now completed. □

Proof of Lemma 3. First, we consider the monotonicity for G_A and G_B. Standard algebraic manipulations yield

$$\frac{\partial}{\partial\psi}G_A(\psi) = \frac{\partial}{\partial\psi}G_B(\psi) = \frac{1-\phi}{\psi^2}\left[e^{\frac{\psi}{1-1/\phi}}\left(1-\frac{\psi}{1-1/\phi}\right)-1\right].$$

Note that $e^a\,(1-a) < 1$ for any $a \in (-\infty,\infty)$. We have, for $\square \in \{A,\,B\}$,

$$\frac{\partial}{\partial\psi}G_\square(\psi) = \begin{cases} <0, & \text{if } \phi<1; \\ >0, & \text{if } \phi>1. \end{cases}$$

Next, let us focus on G_A, and we have

$$\lim_{\psi\to 0} G_A(\psi) = 1-\phi+\lim_{\psi\to 0}\frac{1-\phi}{\psi}(1-e^{\frac{\psi}{1-1/\phi}}) = 1$$

and

$$\lim_{\psi\to\infty} G_A(\psi) = 1-\phi+\lim_{\psi\to\infty}\frac{1-\phi}{\psi}\left(1-e^{\frac{\psi}{1-1/\phi}}\right) = \begin{cases} 1-\phi, & \text{if } \phi<1; \\ \infty, & \text{if } \phi>1. \end{cases}$$

Moreover, note that $G_B(\psi) = G_A(\psi) - (1-\phi)$, the desired results are readily obtained. The proof is completed. □

Proof of Proposition 4. We prove the three relationships one by one. For the first relationship, we know from Equation (26) in the proof of Theorem 1 that $\pi^*_\square(t\,;\psi) = 1/K_\square(t\,;\psi)$ for a fixed $t>0$ and $\square \in \{A,B\}$.

For all $\psi > 0$, we readily obtain

$$\phi \begin{Bmatrix} < \\ > \end{Bmatrix} 1 \quad \Rightarrow \quad G_A(\psi) \begin{Bmatrix} > \\ < \end{Bmatrix} G_B(\psi) \quad \Rightarrow \quad K_A(t\,;\psi) \begin{Bmatrix} < \\ > \end{Bmatrix} K_B(t\,;\psi) \quad \Rightarrow \quad \pi_A^*(t\,;\psi) \begin{Bmatrix} > \\ < \end{Bmatrix} \pi_B^*(t\,;\psi). \tag{27}$$

Collectively, the above inequalities yield

$$\phi \begin{Bmatrix} < \\ > \end{Bmatrix} 1 \quad \Rightarrow \quad c_A^*(0\,;\psi) \begin{Bmatrix} > \\ < \end{Bmatrix} c_B^*(0\,;\psi).$$

Meanwhile, we have

$$\frac{c_A^*(t\,;\psi)}{c_B^*(t\,;\psi)} = \frac{K_B(\psi)}{K_A(\psi)} \times \exp\Big\{ \int_0^t [G_B(\psi) - G_A(\psi)]\lambda_{y+u} du \Big\},$$

which is decreasing in $t > 0$ if $\phi < 1$, or increasing otherwise. This yields the second relationship in the proposition.

Another repeated application of the inequalities in (27) yields the third relationship. This completes the proof. □

Proof of Theorem 5. By definition, the AEW is obtained via solving

$$V_A^*\big(0, x_0\,;\psi\big) = V_B^*\big(0, \mathrm{AEW}\,;\psi\big).$$

According to Theorem 1, the AEW solves

$$\Big[K_A(\psi)\Big]^{-\frac{1-\gamma}{1-\phi}} \frac{x_0^{1-\gamma}}{1-\gamma} = \Big[K_B(\psi)\Big]^{-\frac{1-\gamma}{1-\phi}} \frac{\mathrm{AEW}^{1-\gamma}}{1-\gamma}.$$

This yields

$$\mathrm{AEW}(\psi) = x_0 \left[\frac{K_B(\psi)}{K_A(\psi)}\right]^{1/(1-\phi)},$$

which completes the proof. □

Proof of Proposition 6. From the proof of Proposition 4, we have already known that $\pi^*_\Box(t\,;\psi) = 1/K_\Box(t\,;\psi)$ for a fixed $t > 0$ and $\Box \in \{A, B\}$. Consider

$$\frac{\partial}{\partial \psi} K_\Box(t;\psi) = -\frac{\partial}{\partial \psi} G_\Box(\psi) \times \int_t^\infty \left[\exp\left(-\int_t^s (\beta + G_\Box(\psi)\, \lambda_{y+u}) du \right) \int_t^s \lambda_{y+u} du \right] ds = \begin{cases} > 0, & \text{if } \phi < 1; \\ < 0, & \text{if } \phi > 1. \end{cases}$$

So $\pi^*_\Box(t\,;\psi)$ is decreasing in ψ if the EIS $\phi < 1$, or increasing otherwise. The proof is completed. $\Box$

Proof of Proposition 7. For $t > 0$ and $\Box \in \{A, B\}$, write

$$c^*_\Box(t\,;\psi) = \frac{x_0}{K_\Box(\psi)} \exp\left\{ \int_0^t \big[(1 - G_\Box(\psi))\lambda_{y+u} - \phi(\rho - r) \big] du \right\},$$

so we have

$$\begin{aligned} \frac{\partial}{\partial \psi} c^*_\Box(t\,;\psi) = {} & \frac{x_0}{\big[K_\Box(\psi)\big]^2} \exp\left\{ \int_0^t \big[(1 - G_\Box(\psi))\lambda_{y+u} - \phi(\rho - r) \big] du \right\} \Big(-\frac{\partial}{\partial \psi} G_\Box(\psi) \Big) \\ & \times \left\{ K_\Box(\psi) \int_0^t \lambda_{y+u} du - \int_0^\infty \exp\left(-\int_0^s (\beta + G_\Box(\psi)\, \lambda_{y+u}) du \right) \int_0^s \lambda_{y+u} du\, ds \right\}. \end{aligned}$$

Suppose that $\phi < 1$, from Corollary 2, we know $\frac{\partial}{\partial \psi} G_\Box(\psi) < 0$. An inspection of the partial derivative formula above reveals

$$\lim_{t \to 0} \frac{\partial}{\partial \psi} c^*_\Box(0\,;\psi) < 0 \qquad \text{as well as} \qquad \lim_{t \to \infty} \frac{\partial}{\partial \psi} c^*_\Box(t\,;\psi) > 0.$$

Thereby, $\frac{\partial}{\partial \psi} c^*_\Box(t\,;\psi) = 0$ has an unique root. This establishes the desired results when $\phi < 1$. If $\phi > 1$, then the behavior of the optimal consumption function is reversed. The proof is now completed. $\Box$

Proof of Proposition 8. Recall that by Theorem 5, the AEW can be computed via

$$\text{AEW}(\psi) = x_0 \left[\frac{K_B(\psi)}{K_A(\psi)} \right]^{1/(1-\phi)}, \qquad \psi > 0.$$

To study the monotonicity property for the AEW function, consider

$$\frac{\partial}{\partial \psi}\frac{K_B(\psi)}{K_A(\psi)} = \frac{K_A(\psi)\, K_B'(\psi)\; - K_A'(\psi)\, K_B(\psi)}{K_A(\psi)^2},$$

which has the same sign as $\omega_B(\psi) - \omega_A(\psi)$, where

$$\omega_\square(\psi) = \frac{K_\square'(\psi)}{K_\square(\psi)}, \qquad \square \in \{A, B\}.$$

By letting

$$v(s) = -\frac{\partial}{\partial \psi} G_A(\psi) \times \int_0^s \lambda_{y+u} du, \qquad s > 0,$$

then we can write

$$\omega_\square(\psi) = \int_0^\infty f_\square(s)\, v(s)\, ds = \mathbb{E}\big[v(S_\square)\big],$$

where $S_\square$ has probability density function (PDF):

$$f_\square(s) = \frac{\exp\Big(-\int_0^s (\beta + G_\square(\psi)\, \lambda_{y+u}) du\Big)}{\int_0^\infty \exp\Big(-\int_0^s (\beta + G_\square(\psi)\, \lambda_{y+u}) du\Big) ds}, \qquad s > 0.$$

Note that

$$\frac{f_A(s)}{f_B(s)} = \frac{\int_0^\infty \exp\Big(-\int_0^s (\beta + G_B(\psi)\, \lambda_{y+u}) du\Big) ds}{\int_0^\infty \exp\Big(-\int_0^s (\beta + G_A(\psi)\, \lambda_{y+u}) du\Big) ds} \times \exp\Big\{\int_0^s [G_B(\psi) - G_A(\psi)] \lambda_{y+u} du\Big\},$$

which is decreasing in $s > 0$ if $\phi < 1$, or increasing otherwise. This implies that S_B stochastically dominates (of the first order) S_A if $\phi < 1$, and vice versa. Note that we have already known from Lemma 3, for $\square \in \{A,\, B\}$,

$$\frac{\partial}{\partial \psi} G_\square(\psi) = \begin{cases} < 0, & \text{if } \phi < 1; \\ > 0, & \text{if } \phi > 1. \end{cases}$$

Thereby, $v(s)$ is increasing in $s > 0$ if $\phi < 1$, or decreasing otherwise. Collectively, we can conclude that

$\omega_A(\psi) = \mathbb{E}\big[v(S_A)\big] < \mathbb{E}\big[v(S_B)\big] = \omega_B(\psi)$ for all $\phi > 0$, so $K_B(\psi)/K_A(\psi)$ is increasing in ψ. Thereby, $\mathrm{AEW}(\psi)$ is increasing in ψ when $\phi < 1$, or decreasing otherwise.

The proof is now completed. $\square$

Proof of Proposition 9. The proof can be adapted from that of Proposition 8. We only provide a sketch below. Based on the same formulation as in (18), we specify $K_\square(a)$ as a function of $a = \rho$ or r, $\square \in \{A, B\}$. First, let's consider $\mathrm{AEW}(\rho)$ which can be computed via

$$\mathrm{AEW}(\rho) = x_0 \left[\frac{K_B(\rho)}{K_A(\rho)}\right]^{1/(1-\phi)}, \qquad \rho > 0.$$

Then the monotonicity of $\mathrm{AEW}(\rho)$ is determined by the sign of $\frac{\partial}{\partial \rho}\frac{K_B(\rho)}{K_A(\rho)}$, which is equivalent to that of $\omega_B(\rho) - \omega_A(\rho)$, where

$$\omega_\square(\rho) = \frac{K'_\square(\rho)}{K_\square(\rho)}, \qquad \square \in \{A, B\}.$$

By letting $v(s) = -\phi s$, we can follow essentially the same arguments as in the proof of Proposition 8 to show that $\omega_B(\rho) - \omega_A(\rho) < 0$. Therefore, we can conclude that $\mathrm{AEW}(\rho)$ is decreasing in ρ if $\phi < 1$, or increasing otherwise.

Second, for $\mathrm{AEW}(r)$, we need to define $v(s) = -(1-\phi)s$, which is decreasing if $\phi < 1$, or increasing otherwise. Then, $\omega_B(r) - \omega_A(r) < 0$ if $\phi < 1$, or $\omega_B(r) - \omega_A(r) > 0$ otherwise. That is, if $\phi < 1$ (resp. $\phi > 1$), $\frac{K_B(r)}{K_A(r)}$ is decreasing (resp. increasing) in r. However, the monotonicity of $\mathrm{AEW}(r)$ is also affected by the exponent $1/(1-\phi)$ of $\frac{K_B(r)}{K_A(r)}$. Therefore, in both cases $\phi < 1$ and $\phi > 1$, $\mathrm{AEW}(r) = x_0 \left[\frac{K_B(r)}{K_A(r)}\right]^{1/(1-\phi)}$ is decreasing in r.

The proof is now completed. $\square$

Proof of Proposition 10. For $\square \in \{A, B\}$, function $K_\square(\Delta)$ is associated with the parallelly shifted mortality curve $\{\lambda_{y+t} + \Delta\}_{t \geq 0}$. Note

$$\mathrm{AEW}(\Delta) = x_0 \left[\frac{K_B(\Delta)}{K_A(\Delta)}\right]^{1/(1-\phi)}, \qquad \Delta > 0,$$

whose monotonicity is determined by that of $\frac{K_B(\Delta)}{K_A(\Delta)}$ and the sign of $1-\phi$. To this end, we first check the sign of $\frac{\partial}{\partial\Delta}\frac{K_B(\Delta)}{K_A(\Delta)}$, which is the same as the sign of $\omega_B(\Delta)-\omega_A(\Delta)$, where

$$\omega_\Box(\Delta) = \frac{K'_\Box(\Delta)}{K_\Box(\Delta)}, \qquad \Box \in \{A, B\}.$$

For $s > 0$, letting $v_\Box(s) = -G_\Box(\psi) \times s$, we can write

$$\omega_\Box(\Delta) = \int_0^\infty f_\Box(s)\, v_\Box(s)\, ds = \mathbb{E}\big[v_\Box(S_\Box)\big],$$

where $S_\Box$ has PDF:

$$f_\Box(s) = \frac{\exp\Big(-\int_0^s [\beta + G_\Box(\psi)\,(\lambda_{y+u}+\Delta)]du\Big)}{\int_0^\infty \exp\Big(-\int_0^s [\beta + G_\Box(\psi)\,(\lambda_{y+u}+\Delta)]du\Big)ds}, \qquad s>0, \quad \Box \in \{A, B\}.$$

It remains to compute

$$\omega_B(\Delta)-\omega_A(\Delta) = \mathbb{E}\big[-G_B(\psi)S_B\big] - \mathbb{E}\big[-G_A(\psi)S_A\big] = \mathbb{E}\big[G_A(\psi)S_A\big] - \mathbb{E}\big[G_B(\psi)S_B\big].$$

Hence, unlike the proof of Proposition 8, we next study the first order stochastic dominance between $S_A^* = G_A(\psi)S_A$ and $S_B^* = G_B(\psi)S_B$, whose PDF's can be computed via

$$\begin{aligned} f_\Box^*(s) &= \frac{1}{G_\Box(\psi)} f_\Box\left(\frac{s}{G_\Box(\psi)}\right) \\ &= \frac{\exp\Big(-\int_0^{s/G_\Box(\psi)} \big[\beta + G_\Box(\psi)\,(\lambda_{y+u}+\Delta)\big]du\Big)}{G_\Box(\psi)\int_0^\infty \exp\Big(-\int_0^s \big[\beta + G_\Box(\psi)\,(\lambda_{y+u}+\Delta)\big]du\Big)ds} \\ &= \frac{\exp\Big(-\int_0^s \big[\beta/G_\Box(\psi) + (\lambda_{y+u/G_\Box(\psi)}+\Delta)\big]du\Big)}{G_\Box(\psi)\int_0^\infty \exp\Big(-\int_0^s \big[\beta + G_\Box(\psi)\,(\lambda_{y+u}+\Delta)\big]du\Big)ds}, \qquad s>0, \quad \Box \in \{A, B\}. \end{aligned}$$

Then,

$$\begin{aligned} \frac{f_A^*(s)}{f_B^*(s)} =& \frac{G_B(\psi)\ \int_0^\infty \exp\Big(-\int_0^s \big[\beta + G_B(\psi)\,(\lambda_{y+u}+\Delta)\big]du\Big)ds}{G_A(\psi)\ \int_0^\infty \exp\Big(-\int_0^s \big[\beta + G_A(\psi)\,(\lambda_{y+u}+\Delta)\big]du\Big)ds} \\ & \times \exp\left\{-\int_0^s \Big(\beta\big[1/G_A(\psi) - 1/G_B(\psi)\big] + \big[\lambda_{y+u/G_A(\psi)} - \lambda_{y+u/G_B(\psi)}\big]\Big)du\right\}. \end{aligned} \tag{28}$$

Note that if $\phi < 1$ (resp. $\phi > 1$), then $1/G_A(\psi) - 1/G_B(\psi) < 0$ (resp. > 0) and $\lambda_{y+u/G_A(\psi)} - \lambda_{y+u/G_B(\psi)} < 0$ (resp. > 0), for any $u > 0$. Thus, (28) is increasing in $s > 0$ if $\phi < 1$, or decreasing otherwise.

We can now conclude S_A^* stochastically dominates (of the first order) S_B^*, which implies $\omega_A(\Delta) = \mathbb{E}[-G_A(\psi)S_A] < \mathbb{E}[-G_B(\psi)S_B] = \omega_B(\Delta)$ if $\phi < 1$. The inequality is reversed if $\phi > 1$. By taking into account the sign of $1 - \phi$ when $\phi < 1$ or $\phi > 1$, we readily obtain the desired results in the proposition.

The proof of the proposition is now completed. $\square$