Asymptotic analysis of risk allocation based on the geometric tail expectation risk measure

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Abstract

Risk measure and capital allocation are arguably two of the most important notions in quantitative risk management. They are closely related, as the former often shapes how the latter is implemented. In this paper, we conduct asymptotic analysis of a proportional risk allocation derived using the geometric tail expectation (GTE) risk measure. As a relative and robust variant of the widely adopted conditional tail expectation (CTE) risk measure, the GTE risk measure has garnered increasing attention in recent literature. It induces a proportional allocation that can be regarded as a stochastic counterpart of the deterministic composition that leads to the CTE-induced proportional allocation. Our asymptotic analysis reveals that, across various tail scenarios, the CTE-based and GTE-based allocation methods essentially converge when the confidence level is sufficiently high. This is a welcome finding, since it reconciles discrepancies among risk analysts regarding the selection between the absolute term CTE risk measure and the relative term GTE risk measure for risk allocation purposes. Moreover, the asymptotic equivalence supports advocating for the use of GTE-based allocation as a more versatile alternative to CTE-based allocation, in the sense that the GTE-based allocation is always well-defined while the CTE-based allocation is not. Our simulation example also indicates that under certain data scenarios, the empirical estimator of the GTE-based allocation may exhibit significantly smaller variance compared with that of the CTE-based allocation.

Keywords: Risk analysis, capital allocation, heavy tail, tail dependence, stochastic risk contribution

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1 Introduction

Let $X \geq 0$ be a loss random variable (RV), and the set $\mathcal{X}$ collect all such RV's. Consider a risk portfolio consisting of $n \in \mathbb{N}$ business units (BUs), and let the i-th component of $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{X}^n$ represent the risk associated with the i-th BU, $i \in \mathcal{N} = \{1, \ldots, n\}$. Within the encompassing framework of quantitative risk management, determining the total economic capital needed by a (financial/insurance) conglomerate stands as a centrally important task. The process typically involves quantifying the riskiness associated with the aggregate RV $S = X_1 + \ldots + X_n$ based on a certain risk measure. Another equally important task pertains to the *economic capital allocation*, which revolves around empolying a sensible scheme, typically depicted as relative percentages, for apportioning the total economic capital back to the constituent BUs. An economic capital allocation can be described by a *proportional* allocation principle using the set

$$\left\{ r_i \in [0,1] \right\}_{i=1,\ldots,n} \quad \text{such that } \textstyle\sum_{i=1}^n r_i = 1, \tag{1}$$

where r_i denotes the proportion of risk capital allocated to BU i, and the latter condition ensures that the total capital is fully allocated. Capital allocation carries multiple business purposes, including risk-based pricing, risk and profitability analysis, and risk budgeting (Balog et al., 2017; Chong et al., 2023; Guo et al., 2021; Homburg and Scherpereel, 2008).

Capital allocation can be approached in various ways. Among the myriad methods available, the conditional tail expectation (CTE) based allocation stands out as a prevalent choice among both academics and practitioners. Formally, the CTE-based allocation is formulated as

$$\mathrm{CTE}_q(X_i, S) := E(X_i \,|\, S > s_q), \qquad i \in \mathcal{N},\ q \in [0,1), \tag{2}$$

where $s_q = \inf\left\{ s > 0 : P(S \leq s) > q \right\}$ denotes the value-at-risk (VaR) of S. Beside its practical interpretation, the CTE-based allocation can be justified theoretically via such as Aumann-Shapely allocation (Boonen et al., 2020; Denault, 2001), Euler allocation (Kalkbrener, 2005), distorted allocation (Tsanakas and Barnett, 2003) and weighted allocation (Furman and Zitikis, 2008). Moreover, it is considered optimal according to Laeven and Goovaerts (2004) and Dhaene et al. (2012). Correspondingly, the CTE-based *proportional* allocation is given by

$$r_{i,q} := \frac{\mathrm{CTE}_q(X_i, S)}{\sum_{i=1}^n \mathrm{CTE}_q(X_i, S)} = \frac{\mathrm{CTE}_q(X_i, S)}{\mathrm{CTE}_q(S)}, \qquad i \in \mathcal{N},\ q \in [0,1), \tag{3}$$

where $\mathrm{CTE}_q(X) := \mathrm{CTE}_q(X, X)$, $X \in \mathcal{X}$, denotes the CTE risk measure.

The allocation problem described in equation (1) indicates an inherent connection with the examination of risk composition/contribution (Belles-Sampera et al., 2016; Boonen et al., 2019; Furman et al., 2021). Specifically, let $\boldsymbol{x} = (x_1, \ldots, x_n)$, $s = x_1 + \cdots + x_n$, and denote by $C_i(\boldsymbol{x}) = x_i / s$ the composition function, $i \in \mathcal{N}$. The CTE-based proportional allocation (3) can be regarded as

$$r_{i,q} = C_i\big(\mathrm{CTE}_q(X_1, S), \ldots, \mathrm{CTE}_q(X_n, S)\big) \qquad i \in \mathcal{N},\ q \in [0,1). \tag{4}$$

The formulation in (4) should be viewed as a deterministic composition approach for evaluating risk contribution. A stochastic composition alternative to (4), which is perhaps more natural to interpret from a probabilistic standpoint, can be given by

$$\tilde{r}_{i,q} := \mathrm{CTE}_q(R_i, S) = E(R_i \,|\, S > s_q), \qquad i \in \mathcal{N},\ q \in [0,1), \tag{5}$$

where

$$R_i = C_i(\boldsymbol{X}) = \frac{X_i}{S}, \qquad i \in \mathcal{N}, \tag{6}$$

denotes the relative, stochastic contribution of the risk attributed to the i-th BU (Bauer and Zanjani, 2016; Furman et al., 2021). In contrast to the extensive literature on the CTE risk measure and its associated allocation methods, relatively few results have been established for the aforementioned newly developing allocation rule described in equation (5), which is the primary focus of this paper.

Some comparative discussions are warranted to motivate our study. Allocation rules (3) and (5) are mathematically similar. While the former entails the composition of conditional expectations, the latter relies on the conditional expectation of stochastic risk composition. Altering the order of composition and conditional expectation generally results in different allocation outcomes, although there exist special cases where the two methods produce identical allocations for all $q \in [0,1)$ (Mohammed et al., 2021). It is known that the CTE-based proportional allocation $r_{i,q}$ is a linear approximation of the stochastic composition counterpart $\tilde{r}_{i,q}$ (see Proposition 3 in Mohammed et al., 2021).

Furthermore, akin to the CTE-based allocation, the stochastic composition counterpart (5) also has a sound mathematical justification. Specifically, $\tilde{r}_{i,q}$ can be constructed as an Euler proportional allocation induced by the geometric tail expectation (GTE) risk measure (Bauer and Zanjani, 2016; Mohammed et al., 2021):

$$\mathrm{GTE}_q(X) := \exp\left\{E\left(\log X \,|\, X > x_q\right)\right\}, \qquad X \in \mathcal{X},\ q \in [0,1). \tag{7}$$

Therefore, throughout the remainder of this paper, $\tilde{r}_{i,q}$ defined in (5) will also be referred to as the GTE-based proportional allocation. The GTE risk measure (7) is characterized as a return risk measure according to Bellini et al. (2018), and it is the relative variant of the CTE risk measure which is known as a monetary risk measure (see discussions in Laeven and Gianin, 2022). The adoption of GTE risk measure for capital allocation can be also motivated by the work of Bauer and Zanjani (2016), wherein economic modeling approach is used to reverse engineer the desirable choice of risk measure for capital allocation purpose. Correspondingly, the GTE-based allocation in (5) closely resembles the allocation rule proposed by Bauer and Zanjani (2016) for a profit-maximizing insurer operating in an incomplete market with risk-averse counterparties.

Note that adhering to the "best practice" within the current regulatory environment necessitates a prudent assessment of risk positions within a financial/insurance conglomerate. The confidence level q associated with risk metrics (e.g., VaR, CTE) is usually set close to 1. Consequently, substantial scholarly interest has focused on the asymptotic estimates of risk measures across various risk management contexts (Bassamboo et al., 2008; Cui et al., 2024; Mao et al., 2023; Tang et al., 2019, 2021), as well as their induced capital allocation rules (Asimit and Badescu, 2010; Chen and Liu, 2022, 2024; Hua and Joe, 2011; Qin and Zhou, 2021; Zhu and Li, 2012). This motivates our investigation into the asymptotic behavior of the GTE-based allocation (5) for q approaching 1.

Here is a preview of our research and its findings. We carry out this study under a comprehensive set of assumptions that take into account both heavy-tailed risks and light-tailed ones, both tail dependent risks and tail independent ones. Although we follow the most recent literature to formulate the assumptions via the notion of $\mathbb{M}$ convergence, instead of the traditionally used vague convergence, we show that—apart from relaxed tail indices—they are not much different from the assumptions used in the literature (Asimit et al., 2011). We derive the limits of $\tilde{r}_{i,q}$ as $q \uparrow 1$, for $i \in \mathcal{N}$, and compare the limits of the GTE-based allocations with those of the CTE-based allocations derived in the literature. Remarkably, we find that the two allocation methods (3) and (5) are asymptotically identical under all of the scenarios considered for the risk components.

We believe that our finding is a welcome result to risk analysts. To be specific, we have mentioned earlier that the difference between the CTE-based and GTE-based allocation methods hinges on the selection of either the CTE or GTE risk measures to derive the respective allocation rules. Irrespective of whether one argues from mathematical principles or business fundamentals (e.g., Bauer and Zanjani, 2016; Laeven and Gianin, 2022; Mohammed et al., 2021), the choice of risk measure remains subjective. The asymptotic equivalence elucidated in this study serves to reconcile the discrepancies among risk analysts' divergent perspectives regarding the selection between the CTE and GTE risk measures for risk allocation purposes. This desirable parity between the CTE-based and GTE-based allocations was also studied by Mohammed et al. (2021), where they established the distributional characteristics for the risk set:

$$\mathcal{W} = \left\{ \boldsymbol{X} \in \mathcal{X}^n : \ r_{i,q} = \tilde{r}_{i,q} \text{ for all } i \in \mathcal{N} \text{ and all } q \in [0,1) \right\}.$$

Our current study has shown that relaxing the condition in $\mathcal{W}$ to a more practical scenario, where only q's close to 1 are considered, results in a much broader set of the desirable equivalence.

Further, the asymptotic parity we have obtained provides a strong support for the adaption of the newly developing GTE-based allocation (5) as a more versatile alternative of the CTE-based allocation (3). Firstly, from a probabilistic standpoint, the existence of the CTE-based allocation (3) requires the finitness of the first conditional moment of $\boldsymbol{X}$—for this reason, the literature on the CTE-based allocation for heavy-tailed risks has to restrict to the cases where the the tail indices are greater than 1—whereas the GTE-based allocation (5) is always well defined and is not subject to the restriction. In the realm of insurance risk management, encountering loss distributions having infinite (conditional) means is not uncommon (see discussions in Chen et al., 2024). The GTE-based allocation can be used to deal with such "super heavy-tailed losses", for which the CTE-based allocation is not applicable. Additionally, from a statistical standpoint, when empirical estimators (e.g., Gribkova et al., 2022b) are employed to calculate the allocation ratios, estimating the CTE-based proportional allocation (3) involves two steps: one for estimating $\mathrm{CTE}_q(X_i, S)$ and another for $\mathrm{CTE}_q(S)$. In contrast, computing the GTE-based allocation (5) requires applying an empirical estimator only once. This makes it much more convenient to study, for example, the large sample properties of an empirical GTE allocation estimator for statistical inference. Moreover, our numerical study demonstrates that when the tails of marginal distributions are heavy and tail dependencies are strong, the GTE-based allocation estimator significantly outperforms the CTE-based allocation estimator in terms of lower variations.

The rest of the paper is organized as follows. We start by establishing some standard notations and terminologies in the context of asymptotic analysis in Section 2. In Sections 3 and 4, we study the limit of $\tilde{r}_{i,q}$ under the assumptions of asymptotic dependence and asymptotic independence for $\boldsymbol{X}$, respectively. A simulation study is provided in Section 5 to illustrate the findings obtained from the asymptotic analysis. Finally, Section 6 summarizes the findings and concludes the paper.

2 Preliminaries

This section contains some preliminaries on extreme value theory (EVT) and tail analysis, which will be needed to present the main results. We start by listing our notational convention.

2.1 Notational convention

Throughout this paper, we denote the distribution functions of X_i by F_i and its survival function by $\overline{F}_i$, $i \in \mathcal{N}$. When no confusion could arise, for a positive integer n, we denote $[0,\infty)^n$ by $[\mathbf{0},\boldsymbol{\infty})$ or $\mathbb{R}_+^n$ and $[-\infty,\infty)^n$ by $[-\boldsymbol{\infty},\boldsymbol{\infty})$ or $\mathbb{R}^n$, and likewise, shorthand $(x_1,\infty)\times\ldots\times(x_n,\infty)$ by $(\boldsymbol{x},\boldsymbol{\infty})$, where $\boldsymbol{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$. Vector operations such as $\boldsymbol{x}+\boldsymbol{y}$, $c\boldsymbol{x}$, and $\boldsymbol{x}>\boldsymbol{y}$ for $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$ and $c\in\mathbb{R}$ are to be understood component wise.

Denote by $\mathbb{C}$ a cone in $\mathbb{R}^n$, which, by definition, is closed under positive scale multiplication, and by $\mathbb{C}_0$ a closed cone in $\mathbb{C}$. Denote by $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$ the set of all Borel measures on $\mathbb{C}\setminus\mathbb{C}_0$ that assign finite measure to Borel subsets bounded away from $\mathbb{C}_0$. For a measure $\mu\in\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$ and a measurable set $A\subset\mathbb{C}\setminus\mathbb{C}_0$, we may write $\mu(A)$ as μA when there is no confusion.

All limits are for t tending ∞ unless otherwise stated.

2.2 EVT, regular variation, $\mathbb{M}$ convergence, and tail dependence

Now let us recollect a fundamental result in EVT, the Fisher-Tippett-Gnedenko theorem. It states that if distribution F is in the maximum domain of attraction (MDA) of some non-degenerate distribution G, namely, there exist some $a_m>0$ and $b_m\in\mathbb{R}$, $m=1,2,\ldots$, such that $\lim_{m\to\infty}F^m(a_mx+b_m)=G(x)$, $x\in\mathbb{R}$, then G must be a Fréchet distribution, a Weibull distribution, or a Gumbel distribution. While distributions in the Weibull case have finite upper end points, those in the MDA of the Fréchet and Gumbel distributions may have unbounded right tails. In the context of quantitative risk management, the calculations of economic capital and its allocation primarily rely on the joint behavior of BUs in their right tails, often assumed to be unbounded, particularly for tail risk analysis. Thus, in our paper we are going to focus on the Fréchet and Gumbel cases.

The distributions belonging to the Fréchet case are heavy-tailed and decay at a power rate. Specifically, a distribution function F in the MDA of the Fréchet distribution, written as $F\in\mathrm{MDA}(\Psi)$, is known to have a regularly varying tail, in the sense that

$$\lim_{t\to\infty}\frac{\overline{F}(xt)}{\overline{F}(t)}=x^{-\alpha},\qquad x>0,\tag{8}$$

for some $\alpha>0$. In this case we write $\overline{F}\in\mathrm{RV}_{-\alpha}$. Moreover, if equation (8) holds with $\alpha=\infty$, then we write $\overline{F}\in\mathrm{RV}_{-\infty}$.

A distribution function F with right endpoint $t_F:=\sup\{x\in\mathbb{R}:\overline{F}(x)>0\}$ is said to be in the MDA of the Gumbel distribution, written as $F\in\mathrm{MDA}(\Lambda)$, if and only if there exists a positive function $a(\cdot)$ with $a(t)\to\infty$ and $a(t)=o(t)$, such that,

$$\lim_{t\uparrow t_F}\frac{\overline{F}(t+xa(t))}{\overline{F}(t)}=e^{-x},\qquad x\in\mathbb{R}.\tag{9}$$

In tail risk analysis, we are less concerned with bounded risks and hence in this paper we only focus on the case with $t_F = \infty$ and assume that is the case for every individual risk in portfolio $\boldsymbol{X}$. Note that, the auxiliary function $a(t)$ is self-neglecting, with $\lim_{t\to\infty} a(t+a(t)x)/a(t) = 1$ holding locally uniformly.

To define regular variation for a random vector, we introduce the concept of $\mathbb{M}$ convergence (or, equivalently, $\mathbb{M}^*$ convergence). In general, for a sequence of measures μ and $\{\mu_i\}_{i\geq 1}$ in $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$, we say

$$\mu_n \to \mu \qquad \text{in } \mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0) \tag{10}$$

as $n\to\infty$ if $\lim_{n\to\infty}\int_{\mathbb{C}} f\mathrm{d}\mu_n = \int_{\mathbb{C}} f\mathrm{d}\mu$ holds for any f that is a continuous and bounded function on $\mathbb{C}\setminus\mathbb{C}_0$ with support bounded away from $\mathbb{C}_0$. It is known that the $\mathbb{M}$ convergence in (10) is equivalent to the condition that

$$\lim_{n\to\infty} \mu_n(A) = \mu(A) \tag{11}$$

for every Borel subset A bounded away from $\mathbb{C}_0$ such that the boundary of A is μ negligible; that is, $\mu(\partial A) = 0$. The closed cone $\mathbb{C}_0$ is called a forbidden zone and a region bounded away from $\mathbb{C}_0$ is considered as a tail region. In this paper, we focus on the case with $\mathbb{C}_0 = \{\mathbf{0}\}$ and $\mathbb{C}$ is either $\mathbb{R}^n_+$ or $\mathbb{R}^n$.

Note that there is no standard analogy that characterizes random vectors with marginal distributions in the MDA of the Gumbel distribution through convergence of measures in a metric space. Nonetheless, the structure given in Assumption 2.2 of Asimit et al. (2011) provides a resembling characterization and will be our modeling choice for this paper.

It is well known that tail dependence is a major factor that impacts risk capital allocation. Naturally, we shall consider both asymptotically dependent risks and asymptotically independent risks. A pair of risks X_i and X_j with cumulative distribution functions F_i and F_j respectively, are said to be asymptotically dependent if

$$\liminf_{q\uparrow 1} P\big(F_i(X_i) > q \,|\, F_j(X_j) > q\big) > 0. \tag{12}$$

We use the terms tail dependence and asymptotic dependence interchangeably. Asymptotic dependence for the multivariate cases considered later will entail different forms, but are natural extensions of (12).

3 Asymptotically dependent portfolios

We first consider a portfolio of risks that are asymptotically dependent, exploring two scenarios: one with risks exhibiting Fréchet tails and the other with risks displaying Gumbel tails.

3.1 Fréchet case

For the case where the risks have Fréchet tails, we use the multivariate regular variation (MRV) structure to model the risks. MRV is an integrated structure that models both the marginal distributions with regularly varying tails and their tail dependence. In this paper, we use the notion of $\mathbb{M}$ convergence to define MRV.

($\mathbf{C}_1$) The nonnegative risk vector $\boldsymbol{X}$ possesses MRV; that is, for some nonzero and nondegenerate measure μ, it holds that

$$\frac{P\left(\boldsymbol{X}/t \in \cdot\right)}{\overline{F}_1(t)} \to \mu \qquad \text{in } \mathbb{M}\left(\mathbb{R}_+^n \setminus \{\mathbf{0}\}\right). \tag{13}$$

Moreover, the limit measure μ assigns positive mass to the interior; that is, $\mu(\mathbf{0}, \infty) > 0$.

Note that the condition $\mu(\mathbf{0}, \infty) > 0$ ensures that the risks are asymptotically dependent and, because of the homogeneity of the limit measure μ, is equivalent to the condition that $\mu(\boldsymbol{x}, \infty) > 0$ for some $\boldsymbol{x} > \mathbf{0}$.

It is noteworthy that, traditionally, MRV is defined through vague convergence of Radon measures (Resnick, 2007). However, a recent trend in the literature is to define MRV through $\mathbb{M}$ convergence of measures that are finite on sets bounded away from a designated forbidden zone, which is simply $\{\mathbf{0}\}$ in relation (13). An advantage of formulating MRV using $\mathbb{M}$ convergence is that compactification of the space—which causes problems with polar coordinate transformation—is no longer needed. As an example, to define regular variation of a nonnegative n-dimensional random vector, compactificiation of $[\mathbf{0}, \infty)$ into $[\mathbf{0}, \infty]$ used to be needed so that the Radon measures take finite values on the tail regions; that is, regions bounded away from $\{\mathbf{0}\}$. Such compactification poses problems with, for example, establishing equivalence of vague convergences under Cartesian coordinate and polar coordinate, since polar coordinate transformation such as $\boldsymbol{x} :\mapsto (\|\boldsymbol{x}\|, \boldsymbol{x}/\|\boldsymbol{x}\|)$, where $\|\boldsymbol{x}\|$ denotes a norm of $\boldsymbol{x}$, is only defined on $[\mathbf{0}, \infty) \setminus \{\mathbf{0}\}$ and not on any lines through ∞. Compactification also makes geometric interpretations involving lines through ∞ confusing. The new definition through $\mathbb{M}$ convergence is given by measures that are finite on sets bounded away from $\{\mathbf{0}\}$. The finiteness of the measures on such sets removes the need for compactifying the space. Another advantage of the $\mathbb{M}$ convergence formulation is that MRV as described in ($\mathbf{C}_1$) enables the definition of regular variation on a space with a chosen forbidden zone excluded, thereby facilitating the definition of hidden regular variation on various types of spaces of interest (Das et al., 2013). For applications of hidden regular variation in the context of capital allocation, we refer readers to Hua and Joe (2011).

One may equip the space of $\mathbb{M}\left(\mathbb{R}_+^n \setminus \{\mathbf{0}\}\right)$ with a topology similar to the vague topology (Resnick, 2007, Section 3.3.5), and similar to the vague topology, it is also metrizable. For more discussions about the problems with the traditional definition of MRV, advantages of using $\mathbb{M}$ convergence, how to define a topology for $\mathbb{M}\left(\mathbb{R}_+^n \setminus \{\mathbf{0}\}\right)$, a possible choice of metric for the space, etc., see Das et al. (2013), Lindskog et al. (2014), and Das and Resnick (2017).

In general, to establish $\mathbb{M}$ convergence in $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$, an easy way is to show that the sequence $\{\mu_i\}_{i\geq 1}$ is relatively compact and that convergence holds on a class of convergence-determining sets, usually a π-system. This is easier for the case of regular variation with $\mathbb{C} = \mathbb{R}_+^n$ or $\mathbb{R}^n$ and $\mathbb{C}_0 = \{\mathbf{0}\}$ since relative compactness of the sequence of measures is easier to establish. For example, for regular variation on $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$, relative compactness of the sequence is implied when the convergence of the sequence to the limit measure holds on the determining class of sets that take the form of $A_{\boldsymbol{x}} = [\mathbf{0}, \boldsymbol{x}]^c$, $\boldsymbol{x} \in \mathbb{R}_+^n$. This means that $\mathbb{M}$ convergence in $\mathbb{M}\left(\mathbb{R}_+^n \setminus \{\mathbf{0}\}\right)$ can be demonstrated by proving convergence on all sets of $A_{\boldsymbol{x}}$ given above. Therefore, $\mathbb{M}$ convergence of $\{\mu_i\}_{i\geq 1}$ in $\mathbb{M}\left(\mathbb{R}_+^n \setminus \{\mathbf{0}\}\right)$ is equivalent to vague convergence of $\{\mu_i\}_{i\geq 1}$ in the space of nonnegative Radon measures on $[\mathbf{0}, \infty]^n \setminus \{\mathbf{0}\}$, and hence, at least in the case with $\mathbb{C} = \mathbb{R}_+^n$ or $\mathbb{R}^n$ and $\mathbb{C}_0 = \{\mathbf{0}\}$, the switch from the traditional definition of MRV to the new one is seamless and many results established under vague convergence are still applicable. See, for example, Section 3.1 of Das et al. (2013) for related discussions.

A few comments on condition ($\mathbf{C}_1$) follow. Essentially, the condition is the same as Assumption 2.1 of Asimit et al. (2011), which implies asymptotic dependence among the risk RV's in $\boldsymbol{X}$. It is well known that it also implies $\boldsymbol{X}$ has equivalent tails that are regularly varying. That is, we have $X_i \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$ and every $i \in \mathcal{N}$. Moreover,

$$\lim_{t\to\infty} \frac{\overline{F}_i(t)}{\overline{F}_1(t)} = \lim_{t\to\infty} \frac{P\left(X_i > t, \bigcap_{j\neq i}(X_j \geq 0)\right)}{\overline{F}_1(t)} = \mu\left(\mathbb{R}_+ \times \cdots \times (1,\infty) \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+\right) =: c_i, \tag{14}$$

where $c_i \in (0,\infty)$, $i \in \mathcal{N}$. Using similar reasoning, we can see that the nonnegativity of X_j's is not essential. In fact, the same conclusion would hold as long as there exists a real-valued lower bound for X_j's, which is unsurprising given that we are modeling the right tail behavior.

Theorem 1. *Suppose that the risk vector $\boldsymbol{X}$ satisfies condition ($\mathbf{C}_1$). Then the GTE-based proportional allocation in equation (5) satisfies*

$$\lim_{q\uparrow 1} \tilde{r}_{i,q} = \int_0^1 \frac{\mu(A_{i,z})}{\mu(\aleph)}\, dz, \qquad i \in \mathcal{N}, \tag{15}$$

where $A_{i,z} = \left\{\boldsymbol{x} \in \mathbb{R}_+^n : x_i > z\sum_{k=1}^n x_k, \sum_{k=1}^n x_k > 1\right\}$ and $\aleph = \left\{\boldsymbol{x} \in \mathbb{R}_+^n : \sum_{k=1}^n x_k > 1\right\}$.

Proof. By equation (5), we have, for every $i \in \mathcal{N}$,

$$\lim_{q\uparrow 1} \tilde{r}_{i,q} = \lim_{q\uparrow 1} E(R_i \,|\, S > s_q) = \lim_{t\to\infty} E(R_i \,|\, S > t), \tag{16}$$

where $t = s_q$. Since R_i given by (6) is nonnegative, we have

$$E(R_i \,|\, S > t) = \int_0^1 P(R_i > z \,|\, S > t)\, \mathrm{d}z = \int_0^1 \frac{P(X_i > Sz, S > t)/\overline{F}_1(t)}{P(S > t)/\overline{F}_1(t)}\, \mathrm{d}z. \tag{17}$$

On the one hand, it is well known (Resnick, 2007, Proposition 7.3) that under condition ($\mathbf{C}_1$), $P(S > t) \sim \mu(\aleph)\overline{F}_1(t)$, where $0 < \mu(\aleph) < \infty$ because $\mu \in \mathbb{M}\left(\mathbb{R}_+^n \setminus \{\mathbf{0}\}\right)$.

On the other hand, to examine the numerator in the last term of (17), write $B_z = \{\boldsymbol{x} \in [\mathbf{0},\boldsymbol{\infty}) : x_i = z\sum_{k=1}^n x_k, \sum_{k=1}^n x_k > 1\}$ for $z \in [0,1]$ and $\mathcal{Z} = \{z \in (0,1] : \mu(B_z) > 0\}$; that is, $\mathcal{Z}$ is the collection of points z in $(0,1]$ such that the measure μ assigns a positive mass to the set B_z. We claim that there are at most countably many elements in $\mathcal{Z}$. To see this, for $n = 1,2,\ldots$, write $\mathcal{Z}_n = \{z \in \mathcal{Z} : \mu(B_z) \geq 1/n\}$. If there are uncountably many elements in $\mathcal{Z}$, then, since the union of countably many sets with finite members is at most countable, there exists a positive integer n_0, such that $\mathcal{Z}_{n_0}$ has infinite members, and hence has a countable subset $\mathcal{Z}_{n_0}^c$ with infinite members. Since $B_{z_1} \bigcap B_{z_2} = \emptyset$ for any $z_1 \neq z_2$, this implies $\mu\left(\bigcup_{z\in\mathcal{Z}_{n_0}^c} B_z\right) = \sum_{z\in\mathcal{Z}_{n_0}^c} \mu(B_z) = \infty$. However, noticing that $\bigcup_{z\in[0,1]} B_z = \aleph$ and that B_z's are disjoint, we have $\mu\left(\bigcup_{z\in\mathcal{Z}_{n_0}^c} B_z\right) \leq \mu\left(\bigcup_{z\in[0,1]} B_z\right) = \mu(\aleph) < \infty$, which is a contradiction. Hence, the set $\mathcal{Z}$ has at most countably many members.

For every $z \in (0,1] \setminus \mathcal{Z}$, we have

$$\begin{aligned} \mu(\partial A_{i,z}) &= \mu\left(B_z \bigcup \left\{\boldsymbol{x} \in [\mathbf{0},\boldsymbol{\infty}) : x_i > z\sum_{k=1}^n x_k, \sum_{k=1}^n x_k = 1\right\}\right) \\ &\leq \mu(B_z) + \mu\left(\left\{\boldsymbol{x} \in [\mathbf{0},\boldsymbol{\infty}) : \sum_{k=1}^n x_k = 1\right\}\right) \end{aligned}$$

$$= 0. \tag{18}$$

Hence, it holds for $z \in (0,1] \setminus \mathcal{Z}$ that

$$\begin{aligned}
\lim_{t\to\infty} \frac{P(X_i > Sz, S > t)}{\overline{F}_1(t)} &= \lim_{t\to\infty} \frac{P(X_i/t > Sz/t, S/t > 1)}{\overline{F}_1(t)} \\
&= \mu\left(\boldsymbol{x} \in [\mathbf{0}, \infty) : x_i > z \sum_{k=1}^{n} x_k, \sum_{k=1}^{n} x_k > 1\right) \\
&= \mu(A_{i,z}),
\end{aligned}$$

where in the last step, we used equations (11) and (18). Thus,

$$\begin{aligned}
\lim_{t\to\infty} E(R_i \,|\, S > t) &= \lim_{t\to\infty} \int_{(0,1]\setminus\mathcal{Z}} \frac{P(X_i > Sz, S > t)/\overline{F}_1(t)}{P(S > t)/\overline{F}_1(t)} \mathrm{d}z \\
&= \int_{(0,1]\setminus\mathcal{Z}} \lim_{t\to\infty} \frac{P(X_i > Sz, S > t)/\overline{F}_1(t)}{P(S > t)/\overline{F}_1(t)} \mathrm{d}z \\
&= \int_{(0,1]\setminus\mathcal{Z}} \frac{\mu(A_{i,z})}{\mu(\aleph)} \mathrm{d}z \\
&= \int_0^1 \frac{\mu(A_{i,z})}{\mu(\aleph)} \mathrm{d}z,
\end{aligned}$$

where in the first and last steps we used the fact that $\mathcal{Z}$ has at most countably members, and in the second step we used the Dominated Convergence Theorem. This completes the proof. □

In practical applications, it may be more convenient to work with an asymptotic result expressed in terms of the spectral measure rather than the limit measure. Next, we are going to rewrite our result in Theorem 1 via a semi-parametric form using the spectral measure. To this end, write $\boldsymbol{Y} = \left(1/\overline{F}_1(X_1), \ldots, 1/\overline{F}_n(X_n)\right)$. Under the condition of Theorem 1, there exists a measure μ^*, such that, for any $x > 0$,

$$t P\left(\bigcup_{i=1}^{n} (Y_i > x_i t)\right) \to \mu^*[0, x]^c;$$

see, e.g., Proposition 5.10 of Resnick (2008). For some spectral measure H, which is a probability measure, on $\mathcal{W}_{n-1} = \left\{\boldsymbol{x} \in \mathbb{R}_+^n : \sum_{k=1}^n x_k = 1\right\}$, the limit measure μ^* satisfies $\mathrm{d}\mu^* \circ T^{-1} = n r^{-2} \mathrm{d}r \times \mathrm{d}H$, where T maps $\boldsymbol{x} \in \mathbb{R}_+^n$ into $(r, \boldsymbol{w}) = (\sum_{k=1}^n x_k, x/\sum_{k=1}^n x_k) \in (0,\infty) \times \mathcal{W}_{n-1}$, and T^{-1} denotes its inverse.

Proposition 2. *The asymptotic expression for the GTE-based proportional allocation in Theorem 1 can be rewritten in terms of a spectral measure H as follows:*

$$\lim_{q\uparrow 1} \tilde{r}_{i,q} = \frac{\int_{\mathcal{W}_{n-1}} c_i^{1/\alpha} w_i^{1/\alpha} \left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{\alpha-1} H(\mathrm{d}w)}{\int_{\mathcal{W}_{n-1}} \left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{\alpha} H(\mathrm{d}w)}.$$

Proof. We use the idea of Barbe et al. (2006) to prove the proposition. Write $b_i(\cdot) = \left(1/\overline{F}_i\right)^{\leftarrow}(\cdot)$ and note that, for any $z \in (0,1]$,

$$P(X_i > zS, S > t) = P\left(b_i(Y_i) > z\sum_{k=1}^{n} b_k(Y_k), \sum_{k=1}^{n} b_k(Y_k) > t\right) = P\left(\frac{Y}{s} \in A_{z,t}^*\right),$$

where $s = 1/\overline{F}_1(t)$ and $A_{z,t}^* = \left\{a \in \mathbb{R}_+^n : b_i(a_i s) > z\sum_{k=1}^n b_k(a_k s), \sum_{k=1}^n b_k(a_k s) > t\right\}$. Here we used the facts that $b_i(Y_i) = \left(1/\overline{F}_i\right)^{\leftarrow}\left(1/\overline{F}_i(X_i)\right) = F_i^{\leftarrow}(F_i(X_i))$ and $P(F_i^{\leftarrow}(F_i(X_i)) = X_i) = 1$ (see McNeil et al., 2015, Proposition A.4) and that copulas are invariant under componentwise-monotone increasing transforms. Recall equation (14), which implies that $b_k(s) \sim c_k^{1/\alpha} b_1(s) \sim c_k^{1/\alpha} t$, $k \in N$. It follows that, for any $i \in N$ and $z \in (0,1]$,

$$\frac{P(X_i > zS, S > t)}{\overline{F}_1(t)} = sP\left(\frac{Y}{s} \in A_{z,t}^*\right) \sim sP\left(\frac{Y}{s} \in A_z^*\right) \to \mu^*\left(A_z^*\right),$$

where $A_z^* = \left\{a \in \mathbb{R}_+^n : c_i^{1/\alpha} a_i^{1/\alpha} > z\sum_{k=1}^n c_k^{1/\alpha} a_k^{1/\alpha}, \sum_{k=1}^n c_k^{1/\alpha} a_k^{1/\alpha} > 1\right\}$. The proof of Theorem 1 shows that the left-hand size tends to $\mu(A_z)$, and hence, $\mu(A_z) = \mu^*\left(A_z^*\right)$. Note that

$$\begin{aligned} T^{-1}\left(A_z^*\right) &= \left\{(r, \boldsymbol{w}) \in (0,\infty) \times \mathcal{W}_{n-1} : c_i^{1/\alpha}(rw_i)^{1/\alpha} > z\sum_{k=1}^n c_k^{1/\alpha}(rw_k)^{1/\alpha}, \sum_{k=1}^n c_k^{1/\alpha}(rw_k)^{1/\alpha} > 1\right\} \\ &= \left\{(r, \boldsymbol{w}) \in (0,\infty) \times \mathcal{W}_{n-1} : c_i^{1/\alpha} w_i^{1/\alpha} > z\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}, r > \left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{-\alpha}\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^1 \mu(A_z)\,\mathrm{d}z &= \int_0^1 \mu^*\left(A_z^*\right)\mathrm{d}z \\ &= n\int_0^1 \int_{\mathcal{W}_{n-1}} 1_{\left(c_i^{1/\alpha} w_i^{1/\alpha} > z\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)} \int_{\left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{-\alpha}}^{\infty} r^{-2}\mathrm{d}r H(\mathrm{d}w)\mathrm{d}z \\ &= n\int_0^1 \int_{\mathcal{W}_{n-1}} 1_{\left(c_i^{1/\alpha} w_i^{1/\alpha} > z\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)} \left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{\alpha} H(\mathrm{d}w)\mathrm{d}z \\ &= n\int_{\mathcal{W}_{n-1}} \left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{\alpha} \int_0^1 1_{\left(c_i^{1/\alpha} w_i^{1/\alpha} > z\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)} \mathrm{d}z H(\mathrm{d}w) \\ &= n\int_{\mathcal{W}_{n-1}} \left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{\alpha} \frac{c_i^{1/\alpha} w_i^{1/\alpha}}{\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}} H(\mathrm{d}w) \\ &= n\int_{\mathcal{W}_{n-1}} c_i^{1/\alpha} w_i^{1/\alpha} \left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{\alpha-1} H(\mathrm{d}w). \end{aligned}$$

Similarly, we can show that

$$\mu(\aleph) = n\int_{\mathcal{W}_{n-1}} \left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{\alpha} H(\mathrm{d}w).$$

Together, this leads to an allocation to the ith line of

$$\frac{\int_{\mathcal{W}_{n-1}} c_i^{1/\alpha} w_i^{1/\alpha} \left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{\alpha-1} H(\mathrm{d}w)}{\int_{\mathcal{W}_{n-1}} \left(\sum_{k=1}^n c_k^{1/\alpha} w_k^{1/\alpha}\right)^{\alpha} H(\mathrm{d}w)}, \tag{19}$$

which completes the proof. □

As the succeeding remark will show, despite the distinct expressions between the asymptotic results for the GTE-based allocation, given by (15) in this paper, and the CTE-based allocation, given by Theorem 2.1 of Asimit et al. (2011), a closer examination reveals that they are actually identical. However, an important difference is that the asymptotic result derived for the GTE-based allocation in this paper is applicable to infinite-mean models, whereas the CTE-based allocation in the literature is not. Notably, infinite-mean models have attracted increasing scholarly attention in risk management for areas such as climate risk, catastrophe risk, and operational risk (for recent discussions on infinite-mean models, see, e.g., Chen et al., 2024). This suggests that the GTE-based allocation is a more versatile alternative to the CTE-based allocation widely studied in the existing literature.

Remark 3. *Under condition* ($\mathbf{C}_1$), *Theorem 2.1 of Asimit et al. (2011), together with Proposition 7.3 of Resnick (2007) and equation (9) of Hua and Joe (2011), implies that, if* $\alpha > 1$, *then*

$$\begin{aligned}\lim_{q\uparrow 1} r_{i,q} &= \frac{\alpha-1}{\alpha} \frac{\frac{1}{\alpha-1}\mu\left(\boldsymbol{x}\in\mathbb{R}_+^n : x_i > 1\right) + \int_0^1 \mu\left(\boldsymbol{x}\in\mathbb{R}_+^n : x_i > z, \sum_{k=1}^n x_k > 1\right)\mathrm{d}z}{\mu\left(\boldsymbol{x}\in\mathbb{R}_+^n : \sum_{k=1}^n x_k > 1\right)} \\ &= \frac{c_i + (\alpha-1)\int_0^1 \mu\left(\boldsymbol{x}\in\mathbb{R}_+^n : x_i > z, \sum_{k=1}^n x_k > 1\right)\mathrm{d}z}{\alpha\mu(\aleph)}, \qquad i\in\mathcal{N},\end{aligned}$$

where c_i *is given by equation* (14). *By using a similar idea to that in the proof of Proposition 2 to convert the expression above using the sepctral measure, one can show that the right-hand side of the equation above can also be written as* (19), *and hence,*

$$\lim_{q\uparrow 1} r_{i,q} = \lim_{q\uparrow 1} \tilde{r}_{i,q}, \qquad \textit{for all } i\in\mathcal{N}.$$

The seemingly surprising asymptotic identity above aligns well with our intuition. To see this, note that the decomposition of the limit measure under MRV into a product measure (see Theorem 6.1(4) of Resnick, 2007) shows that the radial component S *and the polar coordinate component* $\boldsymbol{X}/S$ *are independent in the tail. Consequently, for* S *sufficiently large, the conditional covariance between* S *and* R_i, *defined by equation* (6), *diminishes. This implies that* $E(X_i|S>s_q)$ *and* $E(S|S>s_q)\times E(R_i|S>s_q)$ *converge to the same value as* $q\uparrow 1$, *and hence so do* $r_{i,q}$ *and* $\tilde{r}_{i,q}$.

3.2 Gumbel case

In this section, we consider the case where the individual risks follow distributions that are in MDA(Λ). We shall assume the following condition:

($\mathbf{C}_2$) For some positive function $a(\cdot)$ with $a(t) \to \infty$ and $a(t) = o(t)$ and some nonzero and nondegenerate measure μ on $[-\infty,\infty)^n \setminus \{-\boldsymbol{\infty}\}$, the nonnegative risk vector $\boldsymbol{X}$ satisfies that

$$\frac{P((\boldsymbol{X} - t\mathbf{1})/a(t) \in \cdot)}{\overline{F}_1(t)} \to \mu \qquad \text{in } \mathbb{M}([-\infty,\infty)^n \setminus \{-\boldsymbol{\infty}\}), \tag{20}$$

where $\mathbf{1}$ a vector with all components equal to 1 and the limit measure μ satisfies $\mu(-\boldsymbol{\infty},\boldsymbol{\infty}) > 0$.

Condition ($\mathbf{C}_2$) implies that the tails of X_i, $i \in \mathcal{N}$, are equivalent; specifically, $\overline{F}_i(t) \sim c_i\overline{F}_1(t)$, $i \in \mathcal{N}$, with $c_i = \mu(\boldsymbol{x} : x_i > 0)$. Moreover, we remark that, for the limit measure μ in condition ($\mathbf{C}_2$), $\mu\big([-\boldsymbol{\infty},\cdot]^c\big)$ and $\mu\big((\cdot,\boldsymbol{\infty}]\big)$ are continuous functions on $(-\boldsymbol{\infty},\boldsymbol{\infty})$. To see this, it suffices to show that, for small $\delta > 0$ and $\boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^n$ such that $0 < v_i - u_i < \delta$, $i \in \mathcal{N}$, $\mu((\boldsymbol{u},\boldsymbol{v}])$ can be made arbitrarily close to 0. This is obvious since, by equation (41), $\mu((\boldsymbol{u},\boldsymbol{v}]) \le \sum_{i=1}^n (\mu(\boldsymbol{x} : x_i > u_i) - \mu(\boldsymbol{x} : x_i > v_i)) = \sum_{i=1}^n c_i\,(e^{-u_i} - e^{-v_i})$, which goes to 0 as $\delta \to 0$.

The following proposition states that condition ($\mathbf{C}_2$) is equivalent to what is essentially needed in Section 2.2 of Asimit et al. (2011), where their assumption, as described in equation (2.11) of their paper, is formulated in terms of vague convergence. The equivalence hinges on the fact that the limit measure assigns zero mass to the lines through $\boldsymbol{\infty}$. The proof is relegated to the online supplemental material.

Proposition 4. *The $\mathbb{M}$ convergence given in condition ($\mathbf{C}_2$) is equivalent to the existence of some positive auxiliary function $a(\cdot)$ with $a(t) \to \infty$ and $a(t) = o(t)$ and some nonzero and nondegenerate Radon measure ν on $[-\infty,\infty]^n \setminus \{-\boldsymbol{\infty}\}$, such that*

$$\frac{P((\boldsymbol{X} - t\mathbf{1})/a(t) \in \cdot)}{\overline{F}_1(t)} \xrightarrow{v} \nu \qquad \textit{in } \mathbb{M}_+([-\infty,\infty]^n \setminus \{-\boldsymbol{\infty}\}), \tag{21}$$

where $\mathbb{M}_+([-\infty,\infty]^n \setminus \{-\boldsymbol{\infty}\})$ denotes the set of all nonnegative Radon measures on $[-\infty,\infty]^n \setminus \{-\boldsymbol{\infty}\}$.

The equivalence verified in Proposition 4 of this current paper and Note 2.2 of Asimit et al. (2011) allow us to conclude that, under condition ($\mathbf{C}_2$),

$$P(S > nt) \sim \mu\Big(\boldsymbol{x} : \sum_{k=1}^n x_k > 0\Big)\,\overline{F}_1(t). \tag{22}$$

Theorem 5. *Suppose that risk portfolio $\boldsymbol{X}$ has joint distribution satisfying condition ($\mathbf{C}_2$). Then it holds that*

$$\lim_{q\uparrow 1} \tilde{r}_{i,q} = \frac{1}{n}, \qquad i \in \mathcal{N}.$$

Remark 6. *By Theorem 2.2 and equation (2.15) of Asimit et al. (2011), it is obvious that, under condition ($\mathbf{C}_2$), the CTE-based proportional allocation $r_{i,q} \to 1/n$ as $q \uparrow 1$, $i \in \mathcal{N}$, meaning that, asymptotically, it agrees with the GTE-based counterpart.*

Proof of Theorem 5. By equation (5), we have, for every $i \in \mathcal{N}$,

$$\lim_{q\uparrow 1} \tilde{r}_{i,q} = \lim_{q\uparrow 1} E(R_i \,|\, S > s_q) = \lim_{t\to\infty} E(R_i \,|\, S > nt),$$

where $t = s_q/n$. First note that $R_i \geq 0$ and consider

$$\begin{aligned}
E(R_i \,|\, S > nt) &= \int_0^1 P(R_i > z \,|\, S > nt)\,\mathrm{d}z \\
&= \int_0^1 \frac{P(X_i > Sz,\, S > nt)}{P(S > nt)}\,\mathrm{d}z \\
&= \left(\int_0^{1/n} + \int_{1/n}^1\right) \frac{1}{P(S > nt)} P\left(\frac{X_i - t}{a(t)} > z\frac{S - nt}{a(t)} + \frac{(nz-1)t}{a(t)},\, \frac{S - nt}{a(t)} > 0\right) \mathrm{d}z \\
&=: I_1(t) + I_2(t).
\end{aligned}$$

Given any $\varepsilon > 0$ fixed, we have $a(t) < \varepsilon t$ for t sufficiently large. It follows that

$$\begin{aligned}
\limsup_{t\to\infty} I_2(t) &= \limsup_{t\to\infty} \int_{1/n}^1 \frac{1}{P(S > nt)} P\left(\frac{X_i - t}{a(t)} > z\frac{S - nt}{a(t)} + \frac{(nz-1)t}{a(t)},\, \frac{S - nt}{a(t)} > 0\right) \mathrm{d}z \\
&\leq \int_{1/n}^1 \limsup_{t\to\infty} \frac{1}{P(S > nt)} P\Big(\frac{X_i - t}{a(t)} > z\frac{S - nt}{a(t)} + \frac{nz-1}{\varepsilon},\, \frac{S - nt}{a(t)} > 0\Big) \mathrm{d}z \\
&\leq \int_{1/n}^1 \limsup_{t\to\infty} \frac{P\Big((X_i - t)/a(t) > (nz-1)/\varepsilon\Big)/\overline{F}_1(t)}{P(S > nt)/\overline{F}_1(t)}\,\mathrm{d}z \\
&= \int_{1/n}^1 \frac{\mu\Big(\boldsymbol{x} : x_i > (nz-1)/\varepsilon\Big)}{\mu\Big(\boldsymbol{x} : \sum_{k=1}^n x_k > 0\Big)}\mathrm{d}z \\
&= \frac{\int_{1/n}^1 c_i e^{-(nz-1)/\varepsilon}\mathrm{d}z}{\mu\Big(\boldsymbol{x} : \sum_{k=1}^n x_k > 0\Big)} \\
&\leq \frac{c_i \varepsilon}{n\mu\Big(\boldsymbol{x} : \sum_{k=1}^n x_k > 0\Big)},
\end{aligned}$$

where the second step follows from Fatou's lemma and the fact that the integrand on the left-hand size is not greater than 1, the fourth step is due to condition ($\mathbf{C}_2$) and equation (22), and the fifth step arises from equation (41). By the arbitrariness of ε, we have $I_2(t) \to 0$.

Now let us consider $I_1(t)$. It is obvious that $I_1(t) \leq 1/n$ and we now show that $\liminf_{t\to\infty} I_1(t) \geq 1/n$. Note that, for every $z \in (0,1]$,

$$P\left(\frac{X_i - t}{a(t)} > z\frac{S - nt}{a(t)} + \frac{nz-1}{\varepsilon},\, \frac{S - nt}{a(t)} > 0\right) = P\left(\frac{\boldsymbol{X} - t\mathbf{1}}{a(t)} \in A_z\right),$$

where $A_z = \{\boldsymbol{x} \in (-\infty, \infty) : x_i > z\sum_{k=1}^n x_k + (nz-1)/\varepsilon, \sum_{k=1}^n x_k > 0\}$. Write $\mathcal{Z} = \{z \in (0,1] : \mu(\partial A_z) > 0\}$ and $\mathcal{Z}^c = (0,1] \setminus \mathcal{Z}$. Using a similar argument to that in the proof of Theorem 1, we can verify that there are at most countably many elements in $\mathcal{Z}$. Therefore,

$$\begin{aligned}
\liminf_{t\to\infty} I_1(t) &= \liminf_{t\to\infty} \int_{(0,1/n]\cap\mathcal{Z}^c} \frac{1}{P(S > nt)} P\left(\frac{X_i - t}{a(t)} > z\frac{S - nt}{a(t)} + \frac{(nz-1)t}{a(t)},\, \frac{S - nt}{a(t)} > 0\right) \mathrm{d}z \\
&\geq \int_{(0,1/n]\cap\mathcal{Z}^c} \liminf_{t\to\infty} \frac{1}{P(S > nt)} P\left(\frac{X_i - t}{a(t)} > z\frac{S - nt}{a(t)} + \frac{(nz-1)t}{a(t)},\, \frac{S - nt}{a(t)} > 0\right) \mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{(0,1/n]\cap \mathcal{Z}^c} \liminf_{t\to\infty} \frac{P\left((\boldsymbol{X}-t\mathbf{1})/a(t)\in A_z\right)/\overline{F}_1(t)}{P(S>nt)/\overline{F}_1(t)}\,\mathrm{d}z \\
&= \int_{(0,1/n]\cap \mathcal{Z}^c} \frac{\mu(A_z)}{\mu(\boldsymbol{x}:\sum_{k=1}^n x_k>0)}\,\mathrm{d}z \\
&= \int_0^{1/n} \frac{\mu(A_z)}{\mu(\boldsymbol{x}:\sum_{k=1}^n x_k>0)}\,\mathrm{d}z \\
&\geq \frac{1}{n} - \frac{\int_0^{1/n}\mu\left(\boldsymbol{x}:x_i\leq \sum_{k=1}^n x_k+\frac{nz-1}{\varepsilon}\right)\mathrm{d}z}{\mu\left(\boldsymbol{x}:\sum_{k=1}^n x_k>0\right)},
\end{aligned}
\tag{23}
$$

where in the first and fifth steps, we used the fact that $\mathcal{Z}$ has countably many elements, in the second step we used Fatou's lemma, and in the fourth step we used the $\mathbb{M}$ convergence in condition (**C**$_2$), equation (22), and the fact that $\mu(\partial A_z)=0$ for $z\in\mathcal{Z}^c$. Further note that, by equation (41),

$$
\mu\left(\boldsymbol{x}:x_i\leq \sum_{k=1}^n x_k+\frac{nz-1}{\varepsilon}\right)=\mu\left(\boldsymbol{x}:\sum_{k=1,k\neq i}^n x_k\geq -\frac{nz-1}{\varepsilon}\right)\leq \sum_{k=1,k\neq i}^n \mu\left(\boldsymbol{x}:x_k\geq -\frac{nz-1}{(n-1)\varepsilon}\right)\to 0
$$

as $\varepsilon\to 0$. It follows from (23) that $\liminf_{t\to\infty} I_1(t)\geq 1/n$, and hence, we can conclude $\lim_{t\to\infty} I_1(t)=1/n$.

This completes the proof. □

4 Asymptotically independent portfolios

4.1 Fréchet case

The first Fréchet case we consider in this section follows a setup similar to that in Section 3.1, except that we now assume asymptotic independence. Naturally, this leads to a more transparent result regarding the limit of the GTE-based allocation.

(**C**$_3$) Suppose that the nonnegative risk vector $\boldsymbol{X}$ satisfies the $\mathbb{M}$ convergence in condition (**C**$_1$) and that the limit measure $\mu=\mu_I$ which only assigns mass to the axes.

Under condition (**C**$_3$), equation (14) can be rewritten as

$$
\lim_{t\to\infty}\frac{\overline{F}_i(t)}{\overline{F}_1(t)}=\mu_I\left(\mathbb{R}_+\times\cdots\times(1,\infty)\times\mathbb{R}_+\times\cdots\times\mathbb{R}_+\right)=\mu_I\left(\boldsymbol{x}:x_i>1,x_j=0 \text{ for all } j\neq i\right)=:c_i\in[0,\infty).
$$

Note that we also have $\mu_I\left(\boldsymbol{x}\in\mathbb{R}_+^n:x_i>z\right)=c_i z^{-\alpha}$ for $z>0$.

Theorem 7. *Suppose that risk portfolio $\boldsymbol{X}$ satisfies condition* (**C**$_3$)*. Then it holds that*

$$
\lim_{q\uparrow 1}\tilde{r}_{i,q}=\frac{c_i}{\sum_{k=1}^n c_k},\qquad i\in\mathcal{N}. \tag{24}
$$

Proof. Following the proof of Theorem 1, we see that

$$
\lim_{q\uparrow 1}\tilde{r}_{i,q}=\lim_{t\to\infty}E(R_i\,|\,S>t)=\int_0^1\frac{\mu_I(A_{i,z})}{\mu_I(\aleph)}\,\mathrm{d}z,
$$

where $A_{i,z}=\left\{\boldsymbol{x}\in\mathbb{R}_+^n : x_i > z\sum_{k=1}^n x_k, \sum_{k=1}^n x_k>1\right\}$ and $\aleph=\left\{\boldsymbol{x}\in\mathbb{R}_+^n : \sum_{k=1}^n x_k>1\right\}$. Since μ_I puts mass on the axes only, we have

$$\mu_I(\aleph)=\sum_{k=1}^n \mu_I\left(\boldsymbol{x}\in\mathbb{R}_+^n : x_k>1\right)=\sum_{k=1}^n c_k, \tag{25}$$

and for $0<z<1$,

$$\mu_I(A_{i,z})=\mu_I(\boldsymbol{x}\in\mathbb{R}_+^n : x_i>1)=c_i. \tag{26}$$

Combining (25) and (26) yields (24), which completes the proof. □

Remark 8. *Theorem 3.1 of Asimit et al. (2011) and equation (9) of Hua and Joe (2011) imply that, if condition* (**C**$_3$) *holds with tail index* $\alpha>1$, *then* $\lim_{q\uparrow} r_{i,q}=\lim_{q\uparrow 1}\tilde{r}_{i,q}$.

We now present the result for another structure where the components of $\boldsymbol{X}$ are asymptotically independent and belong to $\mathrm{MDA}(\Psi)$. Note that in this case the tail indices of the risks are not necessarily the same.

(**C**$_4$) Suppose that the nonnegative risk vector $\boldsymbol{X}$ has marginal distributions F_i with $\overline{F}_i\in\mathrm{RV}_{-\alpha_i}$, $\alpha_i>0$. Also suppose that there exist some measurable, bounded regularly varying functions $h_i(\cdot):(0,\infty)\to(0,\infty)$, such that, for distinct $i,j\in\mathcal{N}$, the relation $P(X_j>t\,|\,X_i=x)\sim\overline{F}_j(t)\,h_i(x)$ holds uniformly for $x\in[0,\infty)$. In addition, for $n\geq 3$, it is also assumed that for distinct $i,j,k\in\mathcal{N}$, $P(X_j>t, X_k>t\,|\,X_i=x)=o\left(\overline{F}_j(t)+\overline{F}_k(t)\right)h_i(x)$ holds uniformly for $x\in[0,\infty)$.

Condition (**C**$_4$) essentially encapsulates a collection of assumptions made in Assumption 3.2 and Theorem 3.2 of Asimit et al. (2011). It implies asymptotic independence among the risk positions in $\boldsymbol{X}$. The versatility and application of the dependence structure outlined in condition (**C**$_4$) have been discussed extensively in the literature (Asimit and Badescu, 2010; Li et al., 2010).

Under condition (**C**$_4$), it can be shown that, for $i\in\mathcal{N}$,

$$P\left(\sum_{k=1,k\neq i}^n X_k>t\,\middle|\,X_i=x\right)\sim h_i(x)\sum_{k=1,k\neq i}^n \overline{F}_k(t) \tag{27}$$

holds uniformly for $x\in[0,\infty)$. In addition, we have

$$P(S>t)\sim\sum_{k=1}^n \overline{F}_k(t), \tag{28}$$

indicating S has a regularly varying tail with index $\min_{1\leq i\leq n}\alpha_i$. See Lemma 3.2 and equation (3.12) in Theorem 3.2 of Asimit et al. (2011).

The following theorem contains our finding on the GTE-based allocation under condition (**C**$_4$). It's worth noting that the result is formulated in terms of the ratio of tail distribution functions to accommodate different scenarios where tails can be either equivalent or some tails dominate others.

Theorem 9. *Under condition* (C$_4$)*, it holds that*

$$\lim_{q\uparrow 1}\tilde{r}_{i,q}=\lim_{t\to\infty}\frac{\overline{F}_i(t)}{\sum_{k=1}^n\overline{F}_k(t)},\qquad i\in\mathcal{N},\tag{29}$$

provided that the limit on the right-hand side exists.

Remark 10. *If the limit on right-hand side of* (29) *exists, then one may show that it agrees asymptotically with the corresponding CTE-based allocation when condition* (C$_4$) *holds with* $\alpha:=\min_{1\le i\le n}\alpha_i>1$. *To see this, suppose that* $\lim_{t\to\infty}\overline{F}_i(t)/\sum_{k=1}^n\overline{F}_k(t)=r\in[0,1)$*, and note that the CTE-based allocation satisfies*

$$\begin{aligned}\lim_{q\uparrow 1}r_{i,q}&=\lim_{t\to\infty}\frac{E\left(X_ih_i\left(X_i\right)\right)\sum_{k=1,k\neq i}^n\overline{F}_k(t)+t\overline{F}_i(t)\frac{\alpha_i}{\alpha_i-1}}{\sum_{k=1}^n\overline{F}_k(t)E(S|S>t)}\\&=\lim_{t\to\infty}\frac{\alpha-1}{\alpha}\frac{E\left(X_ih_i\left(X_i\right)\right)\sum_{k=1,k\neq i}^n\overline{F}_k(t)+t\overline{F}_i(t)\frac{\alpha_i}{\alpha_i-1}}{t\sum_{k=1}^n\overline{F}_k(t)}\\&=\lim_{t\to\infty}\frac{\alpha-1}{\alpha}\frac{\alpha_i}{\alpha_i-1}\frac{\overline{F}_i(t)}{\sum_{k=1}^n\overline{F}_k(t)},\qquad i\in\mathcal{N}.\end{aligned}$$

Noticing that $\alpha=\alpha_i$ *for* $r>0$*, we conclude that, for every* $r\in[0,1)$*, the above reduces to* r*, which is also the limit of the GTE-based allocation.*

Proof of Theorem 9. First note that equations (16) and (17) remain valid. For $z\in(0,1)$,

$$\begin{aligned}P(X_i>Sz,S>t)&=P(X_i>Sz,S>t,X_i>zt)\\&=P(S>t,X_i>zt)-P(Sz\ge X_i,X_i>zt)\\&=P(S>t,X_i>t)+P(S>t,zt<X_i\le t)-P(Sz\ge X_i,X_i>t)-P(Sz\ge X_i,zt<X_i\le t)\\&=P(X_i>t)+P(S>t,zt<X_i\le t)-P(Sz\ge X_i,X_i>t)-P(Sz\ge X_i,zt<X_i\le t).\end{aligned}$$

Accordingly, we can write

$$\begin{aligned}&\int_0^1P(X_i>Sz,S>t)\mathrm{d}z\\&=P(X_i>t)+\int_0^1P(S>t,zt<X_i\le t)\,\mathrm{d}z-\int_0^1P(Sz\ge X_i,X_i>t)\,\mathrm{d}z-\int_0^1P(Sz\ge X_i,zt<X_i\le t)\,\mathrm{d}z\\&=:I_1(t)+I_2(t)-I_3(t)-I_4(t).\end{aligned}$$

Obviously, $I_4(t)\le I_2(t)$, and hence, by (28), it suffices to show

$$I_2(t)=o(1)P(S>t)\qquad\text{and}\qquad I_3(t)=o(1)P(S>t).$$

We deal with $I_2(t)$ first. Since equation (27) holds uniformly over $x\in[0,\infty)$, for any $\varepsilon>0$ fixed henceforth, there exists $t_0>0$, such that, for all $x>0$ and $t>t_0$,

$$(1-\varepsilon)\,h_i(x)\sum_{k=1,k\neq i}^n\overline{F}_k(t)\;\le\;P\left(S-X_i>t\,|X_i=x\right)\;\le\;(1+\varepsilon)\,h_i(x)\sum_{k=1,k\neq i}^n\overline{F}_k(t).\tag{30}$$

Therefore, we have

$$\begin{aligned}
I_2(t) &= \int_0^1 \int_{zt}^t P\left(S > t \,|X_i = x\right) F_i(\mathrm{d}x)\,\mathrm{d}z \\
&= \int_0^t \frac{x}{t} P\left(S > t \,|X_i = x\right) F_i(\mathrm{d}x) \\
&= \left(\int_0^{t-t_0} + \int_{t-t_0}^t\right) \frac{x}{t} P\left(S > t \,|X_i = x\right) F_i(\mathrm{d}x) \\
&=: I_{21}(t) + I_{22}(t),
\end{aligned}$$

where the second step follows from Fubini's Theorem. Moreover,

$$\begin{aligned}
I_{21}(t) &= \int_0^{t-t_0} \frac{x}{t} P(S - X_i > t - x \,|X_i = x) F_i(\mathrm{d}x) \\
&\le (1+\varepsilon) \sum_{k=1,k\neq i}^{n} \int_0^{t-t_0} \frac{x}{t} h_i(x) \overline{F}_k(t-x) F_i(\mathrm{d}x) \\
&= (1+\varepsilon) \sum_{k=1,k\neq i}^{n} \int_0^{1-t_0/t} \int_{zt}^{t-t_0} h_i(x) \overline{F}_k(t-x) F_i(\mathrm{d}x)\mathrm{d}z \\
&= (1+\varepsilon) \sum_{k=1,k\neq i}^{n} \int_0^{1-t_0/t} \int_{zt}^{t-t_0} \overline{F}_k(t-x) F_i^*(\mathrm{d}x)\mathrm{d}z,
\end{aligned} \tag{31}$$

where in the third step we used Fubini's Theorem, and the distribution function F_i^* introduced in the last step satisfies $F_i^*(\mathrm{d}x) = h_i(x)F_i(\mathrm{d}x)$. Note that F_i^* is a proper distribution function because $E\left(h_i(X_i)\right) = 1$. Since $\overline{F}_i$ and h_i are regularly varying, F_i^* has a regularly varying tail with an index of, say, $-\alpha_i^* < 0$. Moreover, since h_i is bounded, we have

$$\overline{F}_i^*(t) = O\left(\overline{F}_i(t)\right). \tag{32}$$

Let us introduce a nonnegative random variable $X_i^* \sim F_i^*$ that is independent of $\boldsymbol{X}$. It holds for every $z \in (0,1]$ that

$$\begin{aligned}
\int_{zt}^t \overline{F}_k(t-x) F_i^*(\mathrm{d}x) &= P\left(X_i^* + X_k > t, zt < X_i^* \le t\right) \\
&= P\left(X_i^* + X_k > t\right) - P\left(X_i^* > t\right) - P\left(X_i^* + X_k > t, X_i^* \le zt\right) \\
&\le (1+o(1))(\overline{F}_i^*(t) + \overline{F}_k(t)) - \overline{F}_i^*(t) - \overline{F}_k(t) F_i^*(zt) \\
&= o(1)(\overline{F}_i^*(t) + \overline{F}_k(t)),
\end{aligned}$$

where the second last step follows from Lemma 1.3.1 of Embrechts et al. (1997). Therefore,

$$\begin{aligned}
\lim_{t\to\infty} \frac{I_{21}(t)}{P(S>t)} &\le (1+\varepsilon) \sum_{k=1,k\neq i}^{n} \lim_{t\to\infty} \int_0^1 \frac{\int_{zt}^t \overline{F}_k(t-x) F_i^*(\mathrm{d}x)}{P(S>t)} \mathrm{d}z \\
&\le (1+\varepsilon) \sum_{k=1,k\neq i}^{n} \int_0^1 \lim_{t\to\infty} \frac{\int_{zt}^t \overline{F}_k(t-x) F_i^*(\mathrm{d}x)}{P(S>t)} \mathrm{d}z \\
&= 0,
\end{aligned} \tag{33}$$

where the first step is due to equation (31) and the second step follows from the Dominated Convergence Theorem, which applies because, by Lemma 1.3.1 of Embrechts et al. (1997),

$$\frac{\int_{zt}^{t}\overline{F}_k(t-x)F_i^*(\mathrm{d}x)}{P(S>t)} \le \frac{P\left(X_k+X_i^*>t\right)}{P(S>t)} \sim \frac{\overline{F}_k(t)+\overline{F}_i^*(t)}{P(S>t)}.$$

Due to relation (32), the term above is bounded and integrable over $z\in(0,1]$.

Besides,

$$\begin{aligned}
I_{22}(t) &\le \int_{t-t_0}^{t} P(S>t\,|X_i=x)F_i(\mathrm{d}x)\\
&= P\left(S>t,t-t_0<X_i\le t\right)\\
&\le P\left(t-t_0<X_i\le t\right)\\
&= \overline{F}_i(t-t_0)-\overline{F}_i(t)\\
&= o(1)P(S>t).
\end{aligned}$$

The above inequality, together with (33), implies $I_2(t)=o(1)P(S>t)$.

Next, we turn to $I_3(t)$. For any $\delta\in(0,1)$, we have

$$\begin{aligned}
I_3(t) &= \int_0^1\int_t^\infty P\left(S\ge \frac{x}{z}\,\middle|\,X_i=x\right)F_i(\mathrm{d}x)\,\mathrm{d}z\\
&= \left(\int_0^{1-\delta}+\int_{1-\delta}^1\right)\int_t^\infty P\left(S-X_i\ge\left(\frac{1}{z}-1\right)x\,\middle|\,X_i=x\right)F_i(\mathrm{d}x)\,\mathrm{d}z\\
&=: I_{31}(t)+I_{32}(t).
\end{aligned}$$

Let $M=\sup_{1\le i\le n}|h_i|$. For t large, we have

$$\begin{aligned}
I_{31}(t) &\le \int_0^{1-\delta}\int_t^\infty P\left(S-X_i\ge\frac{\delta t}{1-\delta}\,\middle|\,X_i=x\right)F_i(\mathrm{d}x)\,\mathrm{d}z\\
&\le (1+\varepsilon)\sum_{k=1,k\neq i}^{n}\int_0^{1-\delta}\int_t^\infty h_i(x)\overline{F}_k\left(\frac{\delta t}{1-\delta}\right)F_i(\mathrm{d}x)\,\mathrm{d}z\\
&\le (1+\varepsilon)M\overline{F}_i(t)\sum_{k=1,k\neq i}^{n}\overline{F}_k\left(\frac{\delta}{1-\delta}t\right),
\end{aligned}$$

where the second inequality holds due to (30). It is easy to see that $I_{31}(t)=o(1)P(S>t)$. Moreover, for t large,

$$I_{32}(t)=\int_{1-\delta}^1\int_t^\infty P\left(S-X_i\ge\left(\frac{1}{z}-1\right)x\,\middle|\,X_i=x\right)F_i(\mathrm{d}x)\,\mathrm{d}z \;\le\; \delta\overline{F}_i(t).$$

Letting $\delta\to 0$, we obtain $I_3(t)=o(1)P(S>t)$. This completes the proof. □

4.2 Gumbel case

Finally, in this subsection, we consider the case where the marginal distributions of $\boldsymbol{X}$ are in MDA(Λ) and the components are asymptotically independent. Specifically, motivated by Assumption 3.3 of Asimit et al. (2011) (see also Mitra and Resnick, 2009, Section 2.2), we introduce the following condition:[1]

($\mathbf{C}_5$) Suppose that the nonnegative risk vector $\boldsymbol{X}$ has marginal distribution $F_1 \in \mathrm{MDA}(\Lambda)$ with $\overline{F}_i(t)/\overline{F}_1(t) \to c_i$ for some $c_i \in [0,\infty)$, $i \in \mathcal{N}$, and that $\overline{F}_1$ has auxiliary function $a(\cdot)$. Also suppose that, for every $x>0$ and distinct $i,j \in \mathcal{N}$, it holds for some $L_{ij}>0$ that

$$P\left(X_i > t, X_j > a(t)x\right) = o\left(\overline{F}_1(t)\right) \qquad \text{and} \qquad P\left(X_i > L_{ij}a(t), X_j > L_{ij}a(t)\right) = o\left(\overline{F}_1(t)\right). \tag{34}$$

Our main result for this case is given below.

Theorem 11. *Under condition ($\mathbf{C}_5$), it holds that*

$$\lim_{q\uparrow 1} \tilde{r}_{i,q} = \frac{c_i}{\sum_{k=1}^n c_k}, \qquad i \in \mathcal{N}.$$

Remark 12. *With Theorem 11, we can conclude that, by Theorem 3.3 and equation (2.15) of Asimit et al. (2011), the CTE-based and GTE-based allocations agree asymptotically when the risks in $\boldsymbol{X}$ belong to* MDA(Λ) *and satisfy the asymptotic independence condition in ($\mathbf{C}_5$).*

To prove Theorem 11, let us start by considering a simpler two-dimensional case, which will play an important auxiliary role in establishing the desired result in higher dimensions.

Lemma 13. *Let $\mathcal{N} = \{1,2\}$ in condition ($\mathbf{C}_5$). Then*

$$\lim_{q\uparrow 1} \tilde{r}_{i,q} = \frac{c_i}{c_1+c_2}, \qquad i \in \mathcal{N}.$$

Proof. We prove for $i=1$ only and the result for $i=2$ follows immediately. Note that equations (16) and (17) still hold. Obviously,

$$\begin{aligned} E\left(R_1 \mid S>t\right) &= \int_0^1 \frac{P\left(X_1 > zS, S>t\right)}{P\left(S>t\right)}\mathrm{d}z \\ &= \int_0^1 \frac{P\left(X_1 > zX_2/(1-z), X_1+X_2>t\right)}{P\left(S>t\right)}\mathrm{d}z \\ &= \int_0^\infty \frac{1}{(1+w)^2}\frac{P\left(X_1 > wX_2, X_1+X_2>t\right)}{P\left(S>t\right)}\mathrm{d}w, \end{aligned} \tag{35}$$

where the last step follow from a change of variable $w = z/(1-z)$ in the integration.

We claim that the probability $P\left(X_1 > wX_2, X_1+X_2>t\right)$ is asymptotically equivalent to $\overline{F}_1(t)$ for every $w \in (0,\infty)$ and now prove the claim. For every fixed $w>0$, by the proof of Lemma 2.1 of Mitra and Resnick (2009), it

[1]Equation (3.24) in Asimit et al. (2011) contains a minor typo, making it a weaker condition compared to equation (34) included in condition ($\mathbf{C}_5$) of this current paper. However, the assumption used by Asimit et al. (2011) to obtain the desired asymptotics for the CTE-based allocation is exactly equation (34).

holds for some $M > \max\{L_{12}, L_{12}/w\}$ that

$$\frac{1}{\overline{F}_1(t)} P\left(\left(\frac{X_1 - t}{a(t)}, \frac{X_2}{a(t)}\right) \in \cdot\right) \overset{v}{\to} \mu_1 \quad \text{in } \mathbb{M}_+\left([-M,\infty] \times [-\infty,\infty]\right), \tag{36}$$

where $\mathbb{M}_+\left([-M,\infty] \times [-\infty,\infty]\right)$ is the set of all nonnegative Randon measures on $[-M,\infty] \times [-\infty,\infty]$ and the Radon measure μ_1 is defined as $\mu_1(\mathrm{d}x_1, \mathrm{d}x_2) = e^{-x_1}\mathrm{d}x_1 \varepsilon_0(\mathrm{d}x_2)$, where ε_0 denotes the Dirac measure. Now, for fixed $w > 0$ and $M > 0$ given above, split the probability in the numerator of equation (35) as follows:

$$\begin{aligned}
&P(X_1 > wX_2, X_1 + X_2 > t) \\
&= P(X_1 > wX_2, X_1 + X_2 > t, X_1 > t - Ma(t)) + P(X_1 > wX_2, X_1 + X_2 > t, X_1 \le t - Ma(t)) \\
&=: I_1(t) + I_2(t).
\end{aligned}$$

Note that

$$\begin{aligned}
I_2(t) &= P(X_1 > wX_2, X_1 + X_2 > t, X_1 \le t - Ma(t), X_2 > Ma(t)) \\
&\le P(X_1 > wX_2, X_2 > Ma(t)) \\
&\le P(X_1 > L_{12}a(t), X_2 > L_{12}a(t)) \\
&= o\left(\overline{F}_1(t)\right),
\end{aligned}$$

where in the second last step, we used the fact that $M > \max\{L_{12}, L_{12}/w\}$. For $I_1(t)$, write

$$\begin{aligned}
I_1(t) &= P(X_1 > wX_2, X_1 + X_2 > t, X_1 > t - Ma(t), X_2 \le Ma(t)) \\
&\quad + P(X_1 > wX_2, X_1 + X_2 > t, X_1 > t - Ma(t), X_2 > Ma(t)) \\
&= I_{11}(t) + I_{12}(t).
\end{aligned}$$

Obviously, we have $I_{12}(t) \le P(X_1 > wMa(t), X_2 > Ma(t)) = o\left(\overline{F}_1(t)\right)$. Moreover, with $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 > 0, x_1 > -M, x_2 \le M\}$,

$$\begin{aligned}
\lim_{t\to\infty} \frac{I_{11}(t)}{\overline{F_1}(t)} &= \lim_{t\to\infty} \frac{1}{\overline{F_1}(t)} P(X_1 > wX_2, X_1 + X_2 > t, X_1 > t - Ma(t), X_2 \le Ma(t)) \\
&= \lim_{t\to\infty} \frac{1}{\overline{F_1}(t)} P(X_1 + X_2 > t, X_1 > t - Ma(t), X_2 \le Ma(t)) \\
&= \lim_{t\to\infty} \frac{1}{\overline{F_1}(t)} P\left(\left(\frac{X_1 - t}{a(t)}, \frac{X_2}{a(t)}\right) \in A\right) \\
&= \mu_1(A) \\
&= \int_0^\infty e^{-x_1}\mathrm{d}x_1 \\
&= 1,
\end{aligned}$$

where in the second step, we used the fact that $t - Ma(t) \ge wMa(t)$ for t large enough; in the fourth step, we used the vague convergence in equation (36); and in the second last step, we used the fact that μ_1 concentrates on the

x-axis only.

In summary, we have established the claim regarding the asymptotic equivalence between $P(X_1 > wX_2, X_1 + X_2 > t)$ and $\overline{F}_1(t)$ for every $w \in (0, \infty)$. By equation (35), we have

$$
\begin{aligned}
\lim_{q\uparrow 1} \tilde{r}_{1,q} &= \lim_{t\to\infty} E(R_1 | S > t) \\
&= \lim_{t\to\infty} \int_0^\infty \frac{1}{(1+w)^2} \frac{P(X_1 > wX_2, X_1 + X_2 > t)}{P(S > t)} \mathrm{d}w \\
&= \int_0^\infty \frac{1}{(1+w)^2} \lim_{t\to\infty} \frac{P(X_1 > wX_2, X_1 + X_2 > t)}{P(S > t)} \mathrm{d}w \\
&= \frac{c_1}{c_1 + c_2} \int_0^\infty \frac{1}{(1+w)^2} \lim_{t\to\infty} \frac{P(X_1 > wX_2, X_1 + X_2 > t)}{\overline{F_1}(t)} \mathrm{d}w \\
&= \frac{c_1}{c_1 + c_2} \int_0^\infty \frac{1}{(1+w)^2} \mathrm{d}w \\
&= \frac{c_1}{c_1 + c_2},
\end{aligned}
$$

where we used the Dominated Convergence Theorem in the second step. The proof is now completed. □

With the asymptotic result established in the two-dimensional case, we are now in a position to prove the n-dimensional case stated in Theorem 11.

Proof of Theorem 11. Suppose $n > 2$, and write $S_{-i} = S - X_i$. We prove the assertion by showing that S_{-i} and X_i, regarded as two risks, satisfy the conditions for X_1 and X_2 in Lemma 13, respectively.

First, by Corollary 2.2 of Mitra and Resnick (2009),

$$
P(S_{-i} > t) \sim \sum_{k=1, k\neq i}^{n} P(X_k > t) \sim \sum_{k=1, k\neq i}^{n} c_k \overline{F}_1(t),
$$

where $\sum_{k=1,k\neq i}^{n} c_k \geq c_1 = 1$. Hence, because of the closure of MDA(Λ) under tail equivalence, S_{-i} is also in MDA(Λ). This implies the distribution of S_{-i} has an auxiliary function, say, $\tilde{a}(t)$, that is asymptotically equivalent to $a(t)$. Moreover, we obviously have $P(X_i > t)/P(S_{-i} > t) \to c_i / \sum_{k=1,k\neq i}^{n} c_k \geq 0$.

Second, we show that, for every $x > 0$,

$$
P(X_i > x\tilde{a}(t), S_{-i} > t) = o(1) P(S_{-i} > t) \qquad \text{and} \qquad P(X_i > t, S_{-i} > x\tilde{a}(t)) = o(1) P(S_{-i} > t).
$$

The first equation above is a consequence of Lemma 3.4 of Asimit et al. (2011), the asymptotic equivalence between $\tilde{a}(t)$ and $a(t)$, and the tail equivalence between X_1 and S_{-i}. The other equation holds because

$$
\begin{aligned}
\frac{P(X_i > t, S_{-i} > x\tilde{a}(t))}{\overline{F}_1(t)} &\leq \frac{1}{\overline{F}_1(t)} P\left(X_i > t, \bigcup_{k=1,k\neq i}^{n} \left\{ X_k > \frac{x\tilde{a}(t)}{n-1} \right\} \right) \\
&\leq \sum_{k=1,k\neq i}^{n} \frac{1}{\overline{F}_1(t)} P\left(X_i > t, X_k > \frac{x\tilde{a}(t)}{n-1} \right) \\
&\to 0,
\end{aligned}
$$

where the first step is due to the nonnegativity of the risks, and the last step follows from the asymptotic equivalence between $\tilde{a}(t)$ and $a(t)$.

Third, we shall prove that there exists $L_i > 0$, such that

$$P\left(X_i > L_i \tilde{a}(t), S_{-i} > L_i \tilde{a}(t)\right) = o(1) P\left(S_{-i} > t\right). \tag{37}$$

Let $L_i = (n-1)\max_{j,k\in\mathcal{N}, j\neq k} L_{jk}$. We have

$$\begin{aligned}
\frac{P\left(X_i > L_i \tilde{a}(t), S_{-i} > L_i \tilde{a}(t)\right)}{\overline{F}_1(t)} &\leq \frac{P\left(X_i > L_i \tilde{a}(t), \bigcup_{k=1,k\neq i}^{n}\left(X_k > \max_{j,k\in\mathcal{N}, j\neq k} L_{jk}\tilde{a}(t)\right)\right)}{\overline{F}_1(t)} \\
&\leq \sum_{k=1,k\neq i}^{n} \frac{P\left(X_i > L_i \tilde{a}(t), X_k > \max_{j,k\in\mathcal{N}, j\neq k} L_{jk}\tilde{a}(t)\right)}{\overline{F}_1(t)} \\
&\to 0,
\end{aligned}$$

where the first step is due to the nonnegativity of $\boldsymbol{X}$, and the last step is due to condition ($\mathbf{C}_5$) and the asymptotic equivalence between $\tilde{a}(t)$ and $a(t)$. Equation (37) holds since X_1 and S_{-i} are tail equivalent.

Collectively, we have shown that S_{-i} and X_i satisfy all the assumptions in the two-risk case as described in Lemma 13, and thus we readily obtain

$$\lim_{q\uparrow 1} \tilde{r}_{i,q} = \frac{c_i / \sum_{k=1,k\neq i}^{n} c_k}{1 + c_i / \sum_{k=1,k\neq i}^{n} c_k} = \frac{c_i}{\sum_{k=1}^{n} c_k}.$$

This completes the proof. □

5 Numerical illustrations

The numerical study in this section carries two main purposes. Firstly, it illustrates the desirable similarity between the CTE-based allocation and GTE-based allocation when the confidence level q is sufficiently close to 1. Secondly, it demonstrates the smaller variance of the empirical GTE allocation estimator compared to the one associated with the CTE-based allocation under some data scenarios, advocating for the adoption of the GTE-based allocation from a statistical robustness perspective.

The set-up in this section is motivated by the simulation in Asimit et al. (2011). Specifically, we consider a portfolio comprising two dependent risks, denoted as $(X_1, X_2) \in \mathcal{X}^2$. The marginal distributions of X_1 and X_2 follow the Pareto distribution of the second kind (a.k.a., Lomax distribution), with distribution functions:

$$\overline{F}_i(x) = (1 + x/\lambda_i)^{-\alpha} \qquad \text{for } x > 0 \text{ and } i = 1, 2, \tag{38}$$

where $\lambda_i > 0$ and $\alpha > 1$ denote the scale and shape parameters, respectively. The distribution function in (38) belongs to the MDA of the Fréchet distribution. The distribution function $\overline{F}_i$ is regularly varying with a tail index α. The smaller the value of α, the heavier the tails of X_1 and X_2.

Further, the dependence between X_1 and X_2 is assumed to be governed by the Gumbel copula:

$$C(u_1,u_2) = \exp\left\{-\left((-\log u_1)^\beta + (-\log u_2)^\beta\right)^{1/\beta}\right\}, \qquad (u_1,u_2) \in [0,1]^2,\ \beta \geq 1. \tag{39}$$

Figure 1 illustrates the contour plot of the density of the Gumbel copula specified in (39). As depicted, the Gumbel copula exhibits a positive dependence relationship in the upper tail region, with the strength of dependence increasing as β increases. The copula in (39), combined with the marginal distributions in (38), implies that the joint distribution of (X_1, X_2) satisfies condition **(C_1)**.

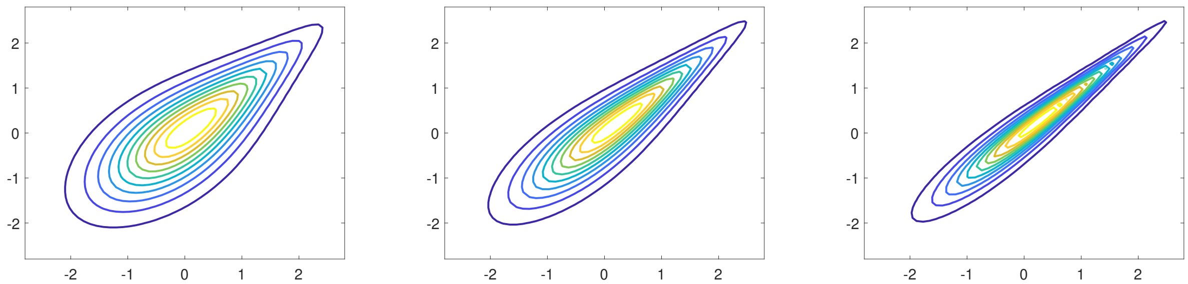

Figure 1: Contour plots of the density function of the Gumbel copula with standard normal margins for $\beta = 2$ in the left panel, $\beta = 3$ in the middle panel, and $\beta = 5$ in the right panel.

With (38) and (39), the limit measure μ in condition **(C_1)** as well as Theorem 1 can be evaluated via (Asimit et al., 2011; Tang and Yuan, 2013):

$$\mu(\boldsymbol{x}, \infty) = x_1^{-\alpha} + b x_2^{-\alpha} - \left(x_1^{-\alpha\beta} + b^\beta x_2^{-\alpha\beta}\right)^{1/\beta},$$

where $b = (\lambda_2/\lambda_1)^\alpha$ and $\boldsymbol{x} \in \mathbb{R}_+^2$. To demonstrate the calculation of the limit in (15), we focus on the limit of $\tilde{r}_{1,q}$ as $q \uparrow 1$, and the limit of $\tilde{r}_{2,q}$ can be obtained via $\lim_{q\uparrow 1} \tilde{r}_{2,q} = 1 - \lim_{q\uparrow 1} \tilde{r}_{1,q}$. Let us rewrite the limit in (15) as

$$\begin{aligned}
&\int_0^1 \frac{\mu\left(\boldsymbol{x} \in [\mathbf{0}, \infty) : x_1 > x_2 z/(1-z),\, x_1 + x_2 > 1\right)}{\mu\left(\boldsymbol{x} \in [\mathbf{0}, \infty) : x_1 + x_2 > 1\right)}\,\mathrm{d}z \\
&= 1 - \int_0^1 \frac{\mu\left(\boldsymbol{x} \in [\mathbf{0}, \infty) : x_2 > x_1 (1-z)/z,\, x_1 + x_2 > 1\right)}{\mu\left(\boldsymbol{x} \in [\mathbf{0}, \infty) : x_1 + x_2 > 1\right)}\,\mathrm{d}z.
\end{aligned} \tag{40}$$

For $\boldsymbol{x} \in \mathbb{R}_+^2$, define

$$\mu_1(\boldsymbol{x}) = -\frac{\partial}{\partial x_1}\mu(\boldsymbol{x}, \infty) = \alpha x_1^{-\alpha-1}\left[1 - \left(x_1^{-\alpha\beta} + b^\beta x_2^{-\alpha\beta}\right)^{1/\beta - 1} x_1^{\alpha(1-\beta)}\right].$$

To compute the denominator in the integration in (40), we have

$$\begin{aligned}
\mu\left(\boldsymbol{x} \in [\mathbf{0}, \infty) : x_1 + x_2 > 1\right) &= \mu\big(\boldsymbol{x} \in [\mathbf{0}, \infty) : x_1 > 1\big) + \mu\big(\boldsymbol{x} \in [\mathbf{0}, \infty) : x_1 \in [0,1], x_1 + x_2 > 1\big) \\
&= 1 + \int_0^1 \mu_1(s, 1-s)\,\mathrm{d}s,
\end{aligned}$$

where the second term is calculated numerically. Now consider the numerator in (40). For $z \in (0,1)$, we have

$$\begin{aligned}
&\mu(\boldsymbol{x} \in [\mathbf{0},\boldsymbol{\infty}) : x_2 > x_1 (1-z)/z, x_1 + x_2 > 1) \\
&= \mu\big(\boldsymbol{x} \in [\mathbf{0},\boldsymbol{\infty}) : x_1 > 1\big) + \mu\big(\boldsymbol{x} \in [\mathbf{0},\boldsymbol{\infty}) : x_1 \in [0,z), x_2 > x_1 (1-z)/z\big) \\
&\quad + \mu\big(\boldsymbol{x} \in [\mathbf{0},\boldsymbol{\infty}) : x_1 \in [z,1], x_2 > 1 - x_1\big) \\
&= 1 + \int_0^z \mu_1(s, s(1-z)/z)\,\mathrm{d}s + \int_z^1 \mu_1(s, 1-s)\,\mathrm{d}s,
\end{aligned}$$

where the last two terms can be computed using numerical integration.

It is noteworthy the dependence structure between X_1 and X_2, described by (39), is exchangeable. Namely, $C(F_1(x_1), F_2(x_2)) = C(F_2(x_2), F_1(x_1))$, for all $x_1, x_2 > 0$. Thereby, if the scale parameters $\lambda_1 = \lambda_2$, then the joint distribution of (X_1, X_2) is exchangeable. In this case, the CTE-based allocation and GTE-based allocations are always identical for any confidence level $q \in [0,1)$, both equal to one half. For this reason, we generally assume $\lambda_1 \neq \lambda_2$ in order to highlight the asymptotic identify established in this current paper.

Inspired by the simulation setup in Asimit et al. (2011), we assume the following parameter values in the baseline scenario:

- The tail index of the marginal distributions: $\alpha = 2$;
- The scale parameters of the marginal distributions: $\lambda_1 = 100\,000$ and $\lambda_2 = 300\,000$;
- The dependence parameter of the Gumbel copula: $\beta = 3$.

For each confidence level $q \in \{0.1, 0.5, 0.8, 0.95\}$, we simulate 10 000 pairs of X_1 and X_2 according to the joint distribution determined by (38) and (39). The empirical tail conditional expectation estimator proposed in Gribkova et al. (2022b) is used to estimate the corresponding CTE-based allocation $r_{i,q}$ and GTE-based allocation $\tilde{r}_{i,q}$, $i = 1,2$. This simulation exercise is repeated 500 times to construct the box plots of the allocation estimates, which are displayed in Figure 2. Note that the adopted tail conditional expectation estimator is already known to be consistent (Gribkova et al., 2022b), so the middle lines in the boxes can be viewed as numeric proxies for the true value of the allocation ratios, while the width of the boxes can be used to assess the robustness of the estimators.

As can be observed, when the confidence level is low (e.g., $q = 0.1$ and $q = 0.5$), the CTE-based and GTE-based allocations differ significantly. At the low confidence level of $q = 0.1$, the black dots are on the boundary or outside the whiskers of the $r_{i,q}$ and $\tilde{r}_{i,q}$ estimates, indicating that the limit provides a poor approximation of the true values of the two allocation ratios. As the confidence level increases to 0.5, the intervals associated with the two allocation methods move closer to the black dots. However, the allocation limit remains outside the whiskers of the GTE-based allocation $\tilde{r}_{i,q}$ estimates. As the confidence level q approaches 1, the medians of the CTE-based and GTE-based allocation estimates converge. At higher confidence levels (i.e., $q = 0.8$ and $q = 0.95$), the allocation limit dots fall within the middle region of the boxes, with the medians approximately matching the allocation ratio limit. Moreover, the intervals associated with the GTE-based allocation estimates are significantly narrower than those of the CTE-based allocation, suggesting that the GTE-based allocation is a statistically robust alternative to the CTE-based allocation when the confidence level is reasonably high.

Next, we proceed by varying the tail parameter α and the copula dependence parameter β to study their impacts on our numerical findings. Since the allocation ratio for the second BU is one minus that for the first BU, we only consider the allocation ratio for the first BU for ease of presentation. Table 1 presents the differences between the

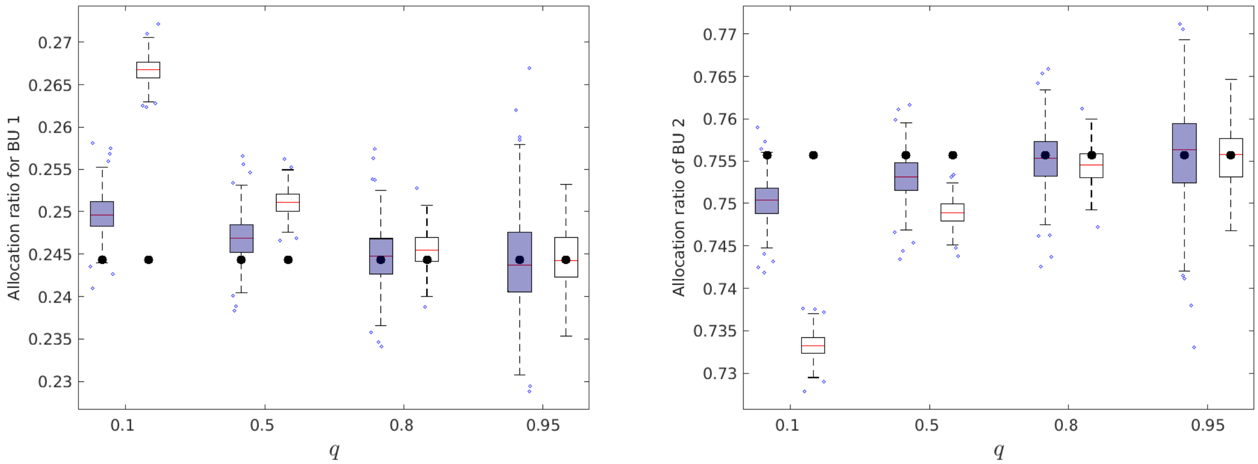

Figure 2: Box plots of the CTE-based allocation estimates (filled boxes) and GTE-based allocation estimates (blank boxes) at varying confidence levels. The black dots represent the limits of the two allocation methods, which are identical.

limits of the CTE-based or GTE-based allocation and the means of the allocation estimates for different values of tail parameter $\alpha \in \{0.8, 2, 3\}$. As mentioned earlier, the empirical estimator used is consistent (Gribkova et al., 2022a,b), so the means can be viewed as reasonable proxies for the true values of the corresponding CTE-based and GTE-based allocations. We observe the followings. Firstly, when $\alpha = 0.8$, the limit of the CTE-based allocation $r_{1,q}$ as $q \uparrow 1$ does not exist, whereas the limit for the GTE-based allocation exists for all $\alpha > 0$. For all values of α considered, the differences between the limits and the (mean approximated) true values become smaller as the confidence level q increases to 1. Secondly, focusing on the high confidence level case (i.e., $q = 0.9$), we find that the difference with the asymptotic approximation for $\alpha = 0.8$ and $\alpha = 2$ is smaller when $\alpha = 2$. This occurs because the magnitudes of the allocation ratios are larger when $\alpha = 0.8$. However, this pattern should not be interpreted as an indication that the asymptotic approximation performs better for risk portfolios with lighter tails. In fact, if we compare the asymptotic approximation difference between $\alpha = 2$ and $\alpha = 3$, where the magnitudes of the allocation ratios are close, we observe the opposite.

	$\alpha = 0.8$		$\alpha = 2$ (Baseline)		$\alpha = 3$	
	CTE method	GTE method	CTE method	GTE method	CTE method	GTE method
Limit	NA	25.60	24.43		24.49	
$q = 0.1$	NA	2.14	0.54	2.24	0.47	1.99
$q = 0.5$	NA	0.84	0.25	0.68	0.12	0.42
$q = 0.8$	NA	0.30	0.04	0.12	0.10	0.06
$q = 0.9$	NA	0.11	0.03	0.02	0.12	0.09

Table 1: Summary of the asymptotic approximation error, defined as the difference between the limit of the CTE-based or GTE-based allocation and the mean of the allocation estimates, for varying values of the tail parameter $\alpha \in \{0.8, 2, 3\}$. All values are reported in the unit of percentage.

Table 2 presents the coefficient of variation for the CTE-based and GTE-based allocation estimates under the same sensitivity analysis settings. As shown, across all tail parameter scenarios considered, the coefficient of variation increases as the confidence level q grows, which is expected as fewer effective samples are used to estimate the allocation ratios. When comparing the CTE and GTE methods, the coefficient of variation is consistently smaller for the empirical GTE-based allocation, except for the case of $q = 1$ and $\alpha = 3$, where they are comparable. In high confidence level and heavy-tailed scenarios (e.g., $q = 0.9$ and $\alpha = 0.8$ or 2), which are statistically challenging yet often encountered in practice, the coefficient of variation associated with the GTE-based allocation estimator is significantly smaller than that of the CTE-based allocation. As a side note, it is worth mentioning that the conditional second moment of X_i given $S > s_q$ is not finite for $\alpha \in (0,2]$. This explains the excessive volatilities associated with the empirical CTE-based allocation when α is low. The above discussion highlights the advantage of adopting the GTE-based allocation from a statistical robustness perspective.

	$\alpha = 0.8$		$\alpha = 2$ (Baseline)		$\alpha = 3$	
	CTE method	GTE method	CTE method	GTE method	CTE method	GTE method
Limit	NA	25.60	24.43		24.49	
$q = 0.1$	28.74	0.64	0.97	0.54	0.50	0.52
$q = 0.5$	28.84	0.83	1.07	0.59	0.55	0.54
$q = 0.8$	29.13	1.32	1.37	0.78	0.72	0.68
$q = 0.9$	29.88	2.73	2.23	1.37	1.19	1.08

Table 2: Summary of the coefficients of variation for the CTE-based and GTE-based allocation estimates for the first BU, with varying tail parameters $\alpha \in \{0.8, 2, 3\}$. All values are reported in the unit of percentage.

A similar sensitivity analysis is conducted for the Gumbel copula's dependence parameter β. According to the limit of allocation ratio shown in Table 3, a smaller value of β, or equivalently, corresponding to a weaker tail dependence, decreases the asymptotic risk allocation to the first BU. Comparing the cases of $\beta = 2$ and $\beta = 3$, the asymptotic approximation error is consistently larger for any considered q when $\beta = 2$, despite the higher corresponding allocation ratio. Although the error is higher when $\beta = 5$ compared to $\beta = 3$, this pattern is likely due to a larger allocation ratio to the BU under $\beta = 5$. Overall, these discussions suggest that a larger value of the dependence parameter β, or, a stronger tail dependence, may improve the performance of the asymptotic approximation.

Table 4 summarizes the changes in the coefficient of variation for allocation estimates in response to varying values of β. The results indicate that the GTE-based allocation estimator consistently outperforms the CTE-based one, exhibiting lower variation across all dependence scenarios considered. With a fixed confidence level, a larger value of β, corresponding to stronger dependence in the copula, enhances the robustness of both the CTE-based and GTE-based allocation estimators.

Collectively, we observe that in this particular simulation example, the empirical GTE-based allocation may outperform the empirical CTE-based allocation in terms of smaller variance in the presence of heavy tails and strong tail dependence. An intuition for this observation is that this distributional scenario implies more frequently occurring extreme values in X_i and S, which tend to occur simultaneously due to strong tail dependence. By taking the conditional mean of the ratio as in the GTE-based allocation, the presence of extremes in X_i and S is

	$\beta = 2$		$\beta = 3$ (Baseline)		$\beta = 5$	
	CTE method	GTE method	CTE method	GTE method	CTE method	GTE method
Limit	23.48		24.43		24.82	
$q = 0.1$	6.24	20.19	0.54	2.24	0.67	3.38
$q = 0.5$	3.68	8.52	0.25	0.68	0.26	0.78
$q = 0.8$	1.12	2.43	0.04	0.12	0.02	0.14
$q = 0.9$	0.10	0.22	0.03	0.02	0.04	0.16

Table 3: Summary of the asymptotic approximation error, defined as the difference between the limit of the CTE-based or GTE-based allocation and the mean of the allocation estimates, for varying values of the Gumbel copula's dependence parameter $\beta \in \{2, 3, 5\}$. All values are reported in the unit of percentage.

	$\beta = 2$		$\beta = 3$ (Baseline)		$\beta = 4$	
	CTE method	GTE method	CTE method	GTE method	CTE method	GTE method
Limit	23.48		24.43		24.82	
$q = 0.1$	1.54	0.72	0.97	0.54	0.57	0.35
$q = 0.5$	1.72	0.88	1.07	0.59	0.62	0.35
$q = 0.8$	2.27	1.27	1.37	0.78	0.79	0.45
$q = 0.9$	3.76	2.30	2.23	1.37	1.27	0.77

Table 4: Summary of the coefficients of variation for the CTE-based and GTE-based allocation estimates for the first BU, with varying tail parameters $\beta \in \{2, 3, 5\}$. All values are reported in the unit of percentage.

directly captured and balanced out, stabilizing the individual ratios and reducing the variance of the estimator. In contrast, the ratio of conditional means, as in the CTE-based allocation, first averages X_i and S separately. These averages tend to smooth out the extreme values to some extent, but the corresponding balancing effect is less efficient compared to directly taking the ratio at the individual level. We openly admit that this observation about the smaller variance associated with the empirical GTE-based allocation is based on a single simulation example. Future research should formalize this observation and investigate the theoretical foundations underlying it.

6 Conclusions

The GTE risk measure is a newly developing, relative, and robust alternative to the widely advocated CTE risk measure. This paper examine the asymptotic behavior of a proportional allocation scheme induced by the GTE risk measure. We consider a variety of asymptotic scenarios, encompassing both tail dependence and tail independence, while accommodating marginal distributions that exhibit either heavy or light tails. We derive the limit of the GTE-based allocation and find that, although for fixed $q \in (0,1)$ it differs from the CTE-based allocation, for q close to 1, they are the same asymptotically under all scenarios considered in this paper. In fact, whether they can differ for $q \uparrow 1$ and under what scenarios they may differ remain some highly nontrivial open questions.

Here are the practical implications of the asymptotic equivalence we have established between the CTE-based and GTE-based allocations. On one hand, if a risk analyst prefers to use the CTE-based allocation, the asymptotic equivalence provides another meaningful perspective for the CTE-based allocation from a profit-maximization standpoint, which motivates the development of the GTE-based allocation (Bauer and Zanjani, 2016). On the other hand, if a risk analyst is open to choosing between the CTE-based and GTE-based allocations, the asymptotic equivalence suggests a favorable consideration for the GTE-based allocation for at least two reasons. Firstly, the CTE-based allocation does not exist for excessively heavy-tailed risks, which are not uncommon in practice, whereas the GTE-based allocation always exists. Secondly, compared to the CTE-based method, the empirical GTE-based estimator may be more robust, exhibiting lower variation particularly when the marginal tails are heavy and tail dependence is strong.

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Supplemental material

This supplemental document contains the proof of Proposition 4 in Section 3.2. For completeness, we recall the proposition before showing its proof.

Proposition 4. *The $\mathbb{M}$ convergence given in condition (C$_2$) is equivalent to the existence of some positive auxiliary function $a(\cdot)$ with $a(t) \to \infty$ and $a(t) = o(t)$ and some nonzero and nondegenerate Radon measure ν on $[-\infty,\infty]^n \setminus \{-\infty\}$, such that*

$$\frac{P((\boldsymbol{X} - t\mathbf{1})/a(t) \in \cdot)}{\overline{F}_1(t)} \xrightarrow{v} \nu \qquad \text{in } \mathbb{M}_+([-\infty,\infty]^n \setminus \{-\infty\}), \tag{21}$$

where $\mathbb{M}_+([-\infty,\infty]^n \setminus \{-\infty\})$ denotes the set of all nonnegative Radon measures on $[-\infty,\infty]^n \setminus \{-\infty\}$.

Proof. The proof is similar to that of Theorem 8.3 of Das et al. (2013) but is adapted for the case with Gumbel marginals.

Write $\mathbb{D} = [-\infty,\infty)^n \setminus \{-\infty\}$ and $\mathbb{E} = [-\infty,\infty]^n \setminus \{-\infty\}$. On the one hand, suppose that the vague convergence in relation (21) holds with auxiliary function $a(\cdot)$ and limit measure ν. We know that the limit measure ν assigns no mass to the lines through ∞. To see this, note that, for every $x_i > -\infty$,

$$\mu([-\infty,\infty] \times \cdots \times (x_i,\infty] \times \cdots \times [-\infty,\infty]) = \lim_{t\to\infty} \frac{P((X_i - t)/a(t) > x_i)}{\overline{F}_1(t)} = c_i e^{-x_i}, \tag{41}$$

where $c_i = \mu(\boldsymbol{x} : x_i > 0)$, and hence, the total mass assigned to the lines through ∞ is not greater than

$$\lim_{x\to\infty} \sum_{i=1}^{n} \mu([-\infty,\infty] \times \cdots \times (x,\infty] \times \cdots \times [-\infty,\infty]) = \lim_{x\to\infty} \sum_{i=1}^{n} c_i e^{-x} = 0.$$

Now define a measure χ_m on $\mathbb{D}$ by $\chi_m(\cdot) = \nu(\cdot)$. Since ν is nonzero and nondegenerate and assigns zero mass to $\mathbb{E} \setminus \mathbb{D}$, we know χ_m is also nonzero and nondegenerate. Moreover, for $A \subset \mathbb{D}$ bounded away from $\{-\infty\}$ with $\chi_m(\partial A) = 0$, it follows from Proposition 6.1 of Resnick (2007) that A is relatively compact in $\mathbb{E}$ with $\nu(\partial A) = 0$. Therefore, relation (21) implies that

$$P((\boldsymbol{X} - t\mathbf{1})/a(t) \in A)/\overline{F}_1(t) \to \nu(A) = \chi_m(A),$$

and hence, by Theorem 2.1 of Lindskog et al. (2014), the $\mathbb{M}$ convergence in (20) holds with limit measure $\mu = \chi_m$ and the same auxiliary function $a(\cdot)$.

On the other hand, suppose that the $\mathbb{M}$ convergence in (20) holds with auxiliary function $a(\cdot)$ and limit measure μ. Define a measure χ_v on $\mathbb{E}$ such that $\chi_v(\cdot) = \mu(\cdot \cap \mathbb{D})$. Since μ is nonzero and nondegenerate, so is χ_v. Now consider an arbitrary set $A \subset \mathbb{E}$ with $\chi_v(\partial A) = 0$ that is relatively compact in $\mathbb{E}$. By Proposition 6.1 of Resnick (2007), A is bounded away from $\{-\infty\}$. Moreover, we have $\mu(\partial(A \cap \mathbb{D})) = \mu(\partial A \cap \mathbb{D}) = \chi_v(\partial A) = 0$. Hence, relation (21) implies that

$$\frac{P((\boldsymbol{X} - t\mathbf{1})/a(t) \in A)}{\overline{F}_1(t)} = \frac{P((\boldsymbol{X} - t\mathbf{1})/a(t) \in A \cap \mathbb{D})}{\overline{F}_1(t)} \to \nu(A \cap \mathbb{D}) = \chi_v(A).$$

This means that the vague convergence given by (21) holds with limit measure $\nu = \chi_\nu$ and the same auxiliary function $a(\cdot)$.

The proof is now complete. $\square$