Lecture

Poisson Processes

Text: A Course in Probability by Weiss 12.1

STAT 225 Introduction to Probability Models
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Average Rate: $\lambda$ per time unit
Number of Successes ($=n$) is distributed $\text{Poisson}(\lambda)$

$I_2 \sim \text{Exp}(\lambda)$  \hspace{1cm}  $I_5 \sim \text{Exp}(\lambda)$

$W_1, \ldots, W_n$ are independently uniformly distributed over $(0,W_n = t]$  

Time between successes is distributed $\text{Exponential}(\lambda)$  

Time intervals are independent if they do not overlap
Example 51

Suppose that phone calls arrive at a switchboard according to a Poisson Process at a rate of 2 per minute.

1. Let $X$ be the number of calls between 9:30 and 9:45. Find the distribution of $X$.

2. Let $T$ be the time between the 8th and 9th calls. What is the distribution (and parameters) of $T$?

3. What is the probability that exactly 10 calls come in the next 4 minutes?

4. What is the probability that the next call comes in 30 seconds and the second call comes at least 45 seconds after that?

5. Given there are exactly 7 calls in 3 minutes, what is the probability that they all came in the last minute?
Example 51 cont’d

Solution.

$X \sim \text{Poisson} (\lambda = 2 \times 15 = 30)$

$T \sim \text{Exp} (\lambda = 2)$

$P(\text{exactly 10 calls come in the next 4 minutes}) = e^{-8} \frac{8^{10}}{10!} = 0.0993$

$P(I_{\text{in}} < 0.5, I_{\text{in}+1} > 0.75) = P(I_{\text{in}} < 0.5) \times P(I_{\text{in}+1} > 0.75) = (1 - e^{-1}) \times e^{-1.5} = 0.6321 \times 0.2231 = 0.1410$

$Z$ be the number of calls came in last minute

$Z \sim \text{Binomial} (n = 7, p = \frac{1}{3})$

$P(Z = 7) = 0.0005$
Example 51 cont’d

Solution.

\[ X \sim \text{Poisson}(\lambda = 2 \times 15 = 30) \]
Example 51 cont’d

Solution.

1. $X \sim \text{Poisson}(\lambda = 2 \times 15 = 30)$
2. $T \sim \text{Exp}(\lambda = 2)$
Example 51 cont’d

Solution.

1. \( X \sim \text{Poisson}(\lambda = 2 \times 15 = 30) \)
2. \( T \sim \text{Exp}(\lambda = 2) \)
3. \( \mathbb{P}(\text{exactly 10 calls come in the next 4 minutes}) = e^{-8} \frac{8^{10}}{10!} = .0993 \)
Example 51 cont’d

Solution.

1. $X \sim Poisson(\lambda = 2 \times 15 = 30)$
2. $T \sim Exp(\lambda = 2)$
3. $P(\text{exactly 10 calls come in the next 4 minutes}) = e^{-8} \frac{8^{10}}{10!} = 0.0993$
4. $P(I_n < .5, I_{n+1} > .75) = P(I_n < .5)P(I_{n+1} > .75) = (1 - e^{-1})(e^{-1.5}) = .6321 \times .2231 = .1410$
Example 51 cont’d

Solution.

1. \( X \sim \text{Poisson}(\lambda = 2 \times 15 = 30) \)

2. \( T \sim \text{Exp}(\lambda = 2) \)

3. \( P(\text{exactly 10 calls come in the next 4 minutes}) = \frac{e^{-8}8^{10}}{10!} = .0993 \)

4. \( P(I_n < .5, I_{n+1} > .75) = P(I_n < .5)P(I_{n+1} > .75) = (1 - e^{-1})(e^{-1.5}) = .6321 \times .2231 = .1410 \)

5. Let \( Z \) be the number of calls came in last minute
   \( Z \sim \text{Binomial}(n = 7, p = \frac{1}{3}) \)
   \( P(Z = 7) = .0005 \)
Example 52

At any point during a Stat 225 exam, the next person to drop a calculator will take 5 minutes on average to do so. Let \( C \) represent the time until the next person drops their calculator.

1. Name the distribution and parameter(s) of \( C \).
2. Find the following probabilities:
   (i) \( P(C > 5 | C < 10) \)
   (ii) \( P(C \geq 8 | C < 15) \)
   (iii) \( C \) is at least 7 given that it is more than 5.
Example 52 cont’d

Solution.

$C \sim \text{Exp}(\lambda = \frac{1}{5})$

(i)

$P(C > 5 | C < 10) = P(5 < C < 10)P(C < 10) = e^{-\frac{5}{1}} - e^{-\frac{2}{1}} = 0.2689$

(ii)

$P(C \geq 8 | C < 15) = P(8 \leq C < 15)P(C < 15) = e^{-\frac{15}{1}} - e^{-\frac{3}{1}} = 0.1601$

(iii)

$P(C \geq 7 | C > 5) = P(C \geq 2) = e^{-\frac{0.4}{1}} = 0.6703$
Example 52 cont’d

Solution.

\[ C \sim Exp(\lambda = \frac{1}{5}) \]

(i) \[ P(C > 5 | C < 10) = P(5 < C < 10) \cdot P(C < 10) = e^{-\frac{1}{5}} - e^{-2} \cdot \frac{1}{5} - e^{-2} = 0.2689. \]

(ii) \[ P(C \geq 8 | C < 15) = P(8 \leq C < 15) \cdot P(C < 15) = e^{-\frac{1}{5} \cdot 6} - e^{-3} \cdot \frac{1}{5} - e^{-3} = 0.1601. \]

(iii) \[ P(C \geq 7 | C > 5) = P(C \geq 2) = e^{-\frac{1}{5} \cdot 4} = 0.6703. \]
Example 52 cont’d

Solution.

\[ C \sim \text{Exp}(\lambda = \frac{1}{5}) \]
Example 52 cont’d

Solution.

1. \( C \sim \text{Exp}(\lambda = \frac{1}{5}) \)

2. (i) \( \mathbb{P}(C > 5 | C < 10) = \frac{\mathbb{P}(5 < C < 10)}{\mathbb{P}(C < 10)} = \frac{e^{-1} - e^{-2}}{1 - e^{-2}} = .2689 \)
Example 52 cont’d

Solution.

1. $C \sim \text{Exp}(\lambda = \frac{1}{5})$  

2. (i) \[ P(C > 5 | C < 10) = \frac{P(5 < C < 10)}{P(C < 10)} = \frac{e^{-1} - e^{-2}}{1 - e^{-2}} = 0.2689 \]

   (ii) \[ P(C \geq 8 | C < 15) = \frac{P(8 \leq C < 15)}{P(C < 15)} = \frac{e^{-1.6} - e^{-3}}{1 - e^{-3}} = 0.1601 \]
Example 52 cont’d

Solution.

1. \( C \sim \text{Exp}(\lambda = \frac{1}{5}) \)

2. 
   (i) \( P(C > 5 | C < 10) = \frac{P(5 < C < 10)}{P(C < 10)} = \frac{e^{-1} - e^{-2}}{1 - e^{-2}} = 0.2689 \)
   
   (ii) \( P(C \geq 8 | C < 15) = \frac{P(8 \leq C < 15)}{P(C < 15)} = \frac{e^{-1.6} - e^{-3}}{1 - e^{-3}} = 0.1601 \)
   
   (iii) \( P(C \geq 7 | C > 5) = P(C \geq 2) = e^{-0.4} = 0.6703 \)