Lecture 7
Independence and Law of Total Probability
Text: A Course in Probability by Weiss 4.3 ~ 4.4

STAT 225 Introduction to Probability Models
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Agenda

1. Quiz 1
2. Independence
3. Law of Total Probability
Quiz 1, Problem 1

Students at Brobbins University were asked to state whether they like chocolate ice cream (yes or no) and whether they like vanilla ice cream (yes or no). The partial results of the poll are below.

- 32% like neither
- 18% like vanilla only
- 15% like both chocolate and vanilla

1. Draw a Venn diagram describing the ice cream flavor preferences at Brobbins University? (5 points)

2. What percent of the students stated that they did not like vanilla at all? (2 points)

3. What percent of those students liking chocolate, did not also like vanilla? (2 points)
Quiz 1, Problem 1 solution

Solution.

By complement rule, we have $1 - \frac{18}{218} = \frac{67}{220} = 67\%$.

Conditional probability: $P(V \cap C) = P(V \cap C) = 0.35.$
Quiz 1, Problem 1 solution

Solution.

\[
\text{By complement rule, we have } 1 - 0.18 - 0.15 = 0.67 = 67\%.
\]

\[
\text{Conditional probability } \Pr(V_c | C) = \frac{\Pr(V_c \cap C)}{\Pr(C)} = \frac{0.35}{0.50} = 0.70 = 70\%.
\]
Solution.

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2. Conditional probability!

$$P(V^c|C) = \frac{P(V^c \cap C)}{P(C)} = \frac{.35}{.50} = 0.70 = 70\%$$
Quiz 1, Problem 2

In a reality television show race there are 12 participants. In one leg of the race the top 3 finishers are immune from elimination and will move on to the next leg. The remaining participants will have to undergo further challenges to be able to move on to the next leg.

1. How many ways can three contestants move to the next round without having to complete further challenges? (3 points)

2. If the first place finisher receives $10000, the second place finisher receives $5000 and third place gets $2500, how many ways can the prize money be awarded to the contestants? (3 points)
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Quiz 1, Problem 2 solution

Solution.

1. \[ \binom{12}{3} = \frac{12!}{9! \times 3!} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = \frac{1320}{6} = 220 \]

2. \[ 12P_3 = \frac{12!}{9!} = 12 \times 11 \times 10 = 1320 \]
Motivating Example

**Example**

You toss a fair coin and it comes up “Heads" three times. What is the chance that the next toss will also be a “Head"?

**Solution.**

The chance is simply $\frac{1}{2}$ just like ANY toss of the coin ⇒ What it did in the past will not affect the current toss!
Independence

Independent events

Let $A$ and $B$ be events of a sample space with $\mathbb{P}(A) > 0$, $\mathbb{P}(B) > 0$. We say that event $B$ is independent of event $A$ if the occurrence of event $A$ does not affect the probability that event $B$ occurs.

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

If we know $A$ and $B$ are independent then the general multiplication rule will reduce to a special form

The special multiplication rule

Two events $A$ and $B$, are said to be independent events if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$$
Example 20

The independence of the events $A$ and $B$ implies that the following are independent:

1. $A^c$ and $B$
2. $A$ and $B^c$
3. $A^c$ and $B^c$
Example 20

Solution.

Given $A$ and $B$ are interdependent
$\Rightarrow P(A|B) = P(A), P(B|A) = P(B)$

1

$P(A^c \cap B) = P(A^c|B) \times P(B) = [1 - P(A|B)] \times P(B)$
$\quad = [1 - P(A)] \times P(B) = P(A^c) \times P(B)$

2

$P(B^c \cap A) = P(B^c|A) \times P(A) = [1 - P(B|A)] \times P(A)$
$\quad = [1 - P(B)] \times P(A) = P(B^c) \times P(A)$

3

$P(A^c \cap B^c) = P(A^c|B^c) \times P(B^c) = [1 - P(A|B^c)] \times P(B^c)$
$\quad = [1 - P(A)] \times P(B^c) = P(A^c) \times P(B^c)$
Independence for more than two events

Consider more than two events $A_1, A_2, \cdots, A_k$, we have two types of independence:

- **Pairwise independent**
  \[
  P(A_i \cap A_j) = P(A_i) \times P(A_j) \quad i, j \in 1, 2, \cdots, k \quad i \neq j
  \]

- **Mutually independent**
  \[
  P(A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_n}) = P(A_{k_1}) \times P(A_{k_2}) \times \cdots \times P(A_{k_n})
  \]

  \[
  k_i, k_j \in 1, 2, \cdots, k \quad k_i \neq k_j \quad 1 \leq n \leq k
  \]
Example 21

Chris and his roommates each have a car. Julia’s Mercedes SLK works with probability .98, Alex’s Mercielago Diablo works with probability .91, and Chris’ 1987 GMC Jimmy works with probability .24. Assume all cars work independently of one another. What is the probability that at least 1 car works?
Example 21: Julia’s SLK
Example 21: Alex’s Diablo
Example 21: Chris’ 1987 Jimmy
Example 21

Solution.

Let $S$ denotes Julia’s SLK works, $M$ denotes Alex’s Mercielago Diablo works, and $J$ denotes Chris’ 1987 GMC Jimmy works.

\[
P(S \cup M \cup J) = 1 - P(S^c \cap M^c \cap J^c) =
1 - P(S^c) \times P(M^c) \times P(J^c) = 1 - .02 \times .09 \times .76 = .9986
\]
Law of total probability

Recall the law of partitions

Let $A_1, A_2, \cdots, A_k$ form a partition of $\Omega$. Then, for all events $B$, 

$$P(B) = \sum_{i=1}^{k} P(A_i \cap B)$$

By applying the general multiplication rule, we have the law of total probability

$$P(B) = \sum_{i=1}^{k} P(B|A_i) \times P(A_i)$$
Example 22

Acme Consumer Goods sells three brands of computers: Mac, Dell, and HP. 30% of the machines they sell are Mac, 50% are Dell, and 20% are HP. Based on past experience Acme executives know that the purchasers of Mac machines will need service repairs with probability .2, Dell machines with probability .15, and HP machines with probability .25. Find the probability a customer will need service repairs on the computer they purchased from Acme.

Solution.

Apply the law of total probability. Let $R$ denotes a customer will need service repairs on the computer, $D$ denotes Dell, $M$ denotes Mac, and $H$ denotes HP.

\[
P(R) = P(R|D) \times P(D) + P(R|M) \times P(M) + P(R|H) \times P(H)
\]

\[
= .5 \times .15 + .3 \times .2 + .2 \times .25 = .185
\]
Example 23

Let us assume that a specific disease is only present in 5 out of every 1,000 people. Suppose that the test for the disease is accurate 99% of the time a person has the disease and 95% of the time that a person lacks the disease. Find the probability that a random person will test positive for this disease.

Solution.

Let $D$ denote disease, and $+$ denote test positive.

Given: $P(D) = .005 \Rightarrow P(D^c) = .995$, $P(+|D) = .99$, $P(-|D^c) = .95 \Rightarrow P(+|D^c) = .05$

$P(+) = P(+|D) \times P(D) + P(+|D^c) \times P(D^c) = .99 \times .005 + .05 \times .995 = .0547$
Summary

In this lecture, we learned

- The concept of Independence
- Special multiplication rule and law of total probability