Conditional Probability and Multiplication Rule

Text: A Course in Probability by Weiss 4.1 ~ 4.2

STAT 225 Introduction to Probability Models
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Conditional Probability

Let $A$ and $B$ be events. The probability that event $B$ occurs given (knowing) that event $A$ occurs is called a conditional probability. It is denoted as $\mathbb{P}(B|A)$. The formula of conditional probability is

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$$

The above formula works so long as $\mathbb{P}(A) > 0$. Under the equally likely framework the formula above can be written as

$$\mathbb{P}(B|A) = \frac{\#(B \cap A)}{\#(A)}$$
General Multiplication Rule

- 2 events:
  \[ P(B \cap A) = P(A) \times P(B|A) \]

- More than 2 events:
  \[
  P(\bigcap_{i=1}^{n} A_i) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_n|A_{n-1} \cap \cdots \cap A_1)
  \]
Example 19

Cheating has been a concern of the dean of the College of Business at Bayview University for several years. Some faculty members in the college believe that cheating is more widespread at Bayview than at other universities, while other faculty members think that cheating is not a major problem in the college. To resolve some of these issues, the dean commissioned a study to assess the current ethical behavior of the business students at Bayview. As a part of this study, an anonymous exit survey was administered to this year’s graduating class. Responses to the following questions were used to obtain data regarding three types of cheating. Any student who answered “Yes” to one or more of these questions was considered to have been involved in some type of cheating.
Example 19 (cont’d)

- During your time at Bayview, did you ever present work copied off the Internet as your own?
- During your time at Bayview, did you ever copy answers off another student’s exam?
- During your time at Bayview, did you ever collaborate with other students on projects that were supposed to be completed individually?

The data are represented in the following Venn diagrams:
Example 19 Venn diagram

Males

Collaborated on Individual projects

Copied off an exam

Copied off the Internet

6 1 0
2 6 1
1

21
Example 19 Venn diagram

Females

Collaborated on Individual projects

Copied off an exam

Copied off the Internet

1 0 3

0 3 3

17
Example 19 Venn diagram

Using the law of partitions, fill in the “Overall” Venn diagram.

Collaborated on Individual projects
Copied off an exam
Copied off the Internet

Overall

7 1 3
5 6 4
5

38
Example 19 (cont’d)

1. What is the probability that a randomly chosen student was involved in some type of cheating? Use the inclusion-exclusion principle, then the idea of complements. Which is simpler?

2. Given that a randomly chosen student cheated, what is the probability that student was male?

3. Given that a randomly chosen student is female, what is the probability that student cheated?

4. What is the probability that a randomly chosen student neither presented work from the Internet nor copied answers off another student’s exam?

5. What is the probability that a randomly chosen student cheated in all three ways, given that the student copied answers off another student’s exam?
Example 19

Solution.

Let $I$ denotes copied off an exam, $P$ denotes collaborated on individual projects, and $E$ denotes copied off the internet.

1. Use the inclusion-exclusion principle:
   \[
   \Pr(I \cup P \cup E) = \Pr(I) + \Pr(P) + \Pr(E) - \Pr(I \cap P) - \Pr(I \cap E) - \Pr(P \cap E) + \Pr(I \cap P \cap E) = \frac{20}{69} + \frac{19}{69} + \frac{13}{69} - \frac{11}{69} - \frac{9}{69} - \frac{6}{69} + \frac{5}{69} = \frac{31}{69}
   \]

2. Use the complement rule: $1 - \frac{38}{69} = \frac{31}{69}$

3. Use the complement rule: $\Pr(M|I \cup P \cup E) = \frac{\Pr(M \cap (I \cup P \cup E))}{\Pr(I \cup P \cup E)} = \frac{\frac{17}{69}}{\frac{31}{69}} = \frac{17}{31}$

4. $\Pr(I \cup P \cup E|F) = \frac{\Pr((I \cup P \cup E) \cap F)}{\Pr(F)} = \frac{\frac{14}{69}}{\frac{31}{69}} = \frac{14}{31}$

5. $\Pr(I^c \cap E^c) = \Pr((I \cup E)^c) = 1 - \Pr(I \cup E) = 1 - \frac{24}{69} = \frac{8}{23}$

6. $\Pr(I \cap P \cap E|E) = \frac{\Pr((I \cap P \cap E) \cap E)}{\Pr(E)} = \frac{\Pr(I \cap P \cap E)}{\Pr(E)} = \frac{\frac{5}{69}}{\frac{13}{69}} = \frac{5}{13}$