Lecture 3
Conditional Probability
Text: A Course in Probability by Weiss 4.1

STAT 225 Introduction to Probability Models
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Agenda

1. Conditional Probability
2. General Multiplication Rule
Motivating Example

In a certain population, the probability a person lives to be 80 is 80% while the probability a person lives to be 90 is 68%. Given that a person lives to be 80, what is the probability that she/he will live to be 90?
Conditional Probability

Let $A$ and $B$ be events. The probability that event $A$ occurs given (knowing) that event $B$ occurs is called a conditional probability. It is denoted as $\mathbb{P}(A|B)$. The formula of conditional probability is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

The above formula works so long as $\mathbb{P}(B) > 0$. Under the equally likely framework the formula above can be written as

$$\mathbb{P}(A|B) = \frac{\#(A \cap B)}{\#(B)}$$
Motivating Example

In a certain population, the probability a person lives to be 80 is 80% while the probability a person lives to be 90 is 68%. Given that a person lives to be 80, what is the probability that she/he will live to be 90?

Solution.

- Event $A$: a person lives to be 90
- Event $B$: a person lives to be 80

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\text{a person lives to be 80 AND a person lives to be 90})}{P(\text{a person lives to be 80})} = \frac{P(\text{a person lives to be 90})}{P(\text{a person lives to be 80})}$$

$$= \frac{0.68}{0.80} = 0.85$$
Given B, what’s the probability of A?

In a conditional probability problem, the sample space is “reduced” to the “space” of the given outcome (e.g. if given B, we now just care about the probability of A occurring “inside” of B).
General Multiplication Rule

Suppose we know the conditional probability \( P(A|B) \) and the marginal probability i.e. the probability of the given event \( P(B) \). Then the formula of conditional probability provides a way to compute the joint probability \( P(A \cap B) \)

- 2 events:
  \[
P(A \cap B) = P(B) \times P(A|B)
  \]

- More than 2 events:
  \[
P(\cap_{i=1}^{n} A_i) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_n|A_{n-1} \cap \cdots \cap A_1)
  \]
Example 11

A Morgan Stanley Consumer Research Survey sampled men and women and asked each whether they preferred to drink plain bottled water or a sports drink such as Gatorade or Propel Fitness water (*The Atlanta Journal-Constitution, December 28, 2005*). Suppose 200 men and 200 women participated in the study, and 280 reported they preferred plain bottled water. Of the group preferring a sports drink, 80 were men and 40 were women. Let

- \( M \): the event the consumer is a man
- \( W \): the event the consumer is a woman
- \( B \): the event the consumer preferred plain bottled water
- \( S \): the event the consumer preferred a sports drink
Example 11 (cont’d)

Answer the following:

1. What is the probability a person in the study preferred plain bottled water?
2. What is the probability a person in the study preferred a sports drink?
3. What are the conditional probabilities $P(M|S)$ and $P(W|S)$?
4. What are the joint probabilities $P(M \cap S)$ and $P(W \cap S)$?
5. Given a consumer is a man, what is the probability he will prefer a sports drink?
Example 11

Solution.

1. $P(B) = \frac{280}{400} = 0.7$
2. $P(S) = \frac{120}{400} = 0.3$
3. $P(M|S) = \frac{P(M \cap S)}{P(S)} = \frac{80}{120} = \frac{2}{3}, \quad P(W|S) = \frac{P(W \cap S)}{P(S)} = \frac{40}{120} = \frac{1}{3}$
4. $P(M \cap S) = P(S) \times P(M|S) = 0.3 \times \frac{2}{3} = 0.2, \quad P(W \cap S) = P(S) \times P(W|S) = 0.3 \times \frac{1}{3} = 0.1$
5. $P(S|M) = \frac{P(S \cap M)}{P(M)} = \frac{80}{200} = 0.4$
Example 12 (Example 10 revisit)

Using the Venn Diagram summarizing the distribution of operating systems previously described, calculate the following:

1. The probability that a randomly chosen student uses all three operating systems, given the student uses Windows
2. The probability that a randomly chosen student uses all three operating systems, given the student does not use Windows
3. The probability that a randomly chosen student uses Windows, given the student uses Mac OS
4. The probability that a randomly chosen student does not use any of the operating systems, given the student does not use Windows
Example 12 Venn Diagram

Linux

Mac OS

Windows

9 3 2

17

4 2 9

2
Example 12

Solution.

\[ P(W \cap M \cap L | W) = \frac{P((W \cap M \cap L) \cap W)}{P(W)} = \frac{P(W \cap M \cap L)}{P(W)} = \frac{\frac{9}{50}}{\frac{30}{50}} = 0.3 \]

\[ P(W \cap M \cap L | W^c) = \frac{P((W \cap M \cap L) \cap W^c)}{P(W^c)} = \frac{P(\emptyset)}{P(W^c)} = \frac{0}{\frac{20}{50}} = 0 \]

\[ P(W | M) = \frac{P(W \cap M)}{P(M)} = \frac{\frac{11}{50}}{\frac{18}{50}} = \frac{11}{18} \]

\[ P((W \cup M \cup L)^c | W^c) = \frac{P((W \cup M \cup L)^c \cap W^c)}{P(W^c)} = \frac{P((W \cup M \cup L)^c)}{P(W^c)} = \frac{1 - \frac{46}{50}}{\frac{20}{50}} = 0.2 \]
Summary

In this lecture, we learned

- Conditional probability: definition, formula, venn diagram representation
- General multiplication rule