Lecture 10
Expected Value and Variance

Text: A Course in Probability by Weiss 7.1 ∼ 7.3

STAT 225 Introduction to Probability Models
February 9, 2014

Whitney Huang
Purdue University
Agenda

1. Random Variable (r.v.)
2. Expected Value
3. Variance
A random variable is a real–valued function whose domain is the sample space of a random experiment. In other words, a random variable is a function $X : \Omega \mapsto \mathbb{R}$, where $\Omega$ is the sample space of the random experiment under consideration and $\mathbb{R}$ represents the set of all real numbers.
**Expected Value**

The expected value of a discrete r.v. $X$, denoted by $\mathbb{E}[X]$ is defined by

$$\mathbb{E}[X] = \sum_x x \times p_X(x)$$

**Remark:** The expected value of a discrete r.v. is a weighted average of its possible values, and the weight used is its probability. Sometimes we refer to the expected value as the expectation, the mean, or the first moment. It is usually denoted by $\mu_X$

For any function, say $g(X)$, we can also find an expectation of that function. It is

$$\mathbb{E}[g(X)] = \sum_x g(x) \times p_X(x)$$

Ex

$$\mathbb{E}[X^2] = \sum_x x^2 \times p_X(x)$$
Properties of expected value

Let $X$ and $Y$ be discrete r.v.s defined on the same sample space and having finite expectation (i.e. $\mathbb{E}[X], \mathbb{E}[Y] < \infty$). Let $a$ and $b$ be constants. Then the following hold:
Properties of expected value

Let $X$ and $Y$ be discrete r.v.s defined on the same sample space and having finite expectation (i.e. $E[X], E[Y] < \infty$). Let $a$ and $b$ be constants. Then the following hold:

- $E[X + Y] = E[X] + E[Y]$
Properties of expected value

Let $X$ and $Y$ be discrete r.v.s defined on the same sample space and having finite expectation (i.e. $E[X], E[Y] < \infty$). Let $a$ and $b$ be constants. Then the following hold:

- $E[X + Y] = E[X] + E[Y]
- E[aX + b] = a \times E[X] + b$
Example 25 revisited

The following is a chart describing the number of siblings each student in a particular class has.

<table>
<thead>
<tr>
<th>Siblings (X)</th>
<th>Frequency</th>
<th>Relative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
<td>.200</td>
</tr>
<tr>
<td>1</td>
<td>17</td>
<td>.425</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>.275</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>.075</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>.025</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>40</strong></td>
<td><strong>1</strong></td>
</tr>
</tbody>
</table>

Find the expected value of the number of siblings

Solution.

\[
\mathbb{E}[X] = \sum_x x p_X(x) = 0 \times .200 + 1 \times .425 + 2 \times .275 + 3 \times .075 + 4 \times .025 = 1.3
\]
**Variance**

The variance of a r.v. is a measure of the spread, or variability, in the r.v. The variance of a discrete r.v. $X$, denoted by $Var(X)$, is defined by

$$Var(X) = E[(X - \mu_x)^2]$$

or

$$Var(X) = E[X^2] - (E[X])^2$$
Properties of variance

Let $c$ be a constant. Then the following hold:
Properties of variance

Let $c$ be a constant. Then the following hold:

- $\text{Var}(cX) = c^2 \times \text{Var}(X)$
Properties of variance

Let $c$ be a constant. Then the following hold:

1. $\text{Var}(cX) = c^2 \times \text{Var}(X)$
2. $\text{Var}(X + c) = \text{Var}(X)$
Example 26

Suppose $X$ and $Y$ are random variables with $\mathbb{E}[X] = 3$, $\mathbb{E}[Y] = 4$ and $\text{Var}(X) = 4$. Find:

1. $\mathbb{E}[2X + 1]$
2. $\mathbb{E}[X - Y]$
3. $\mathbb{E}[X^2]$
4. $\mathbb{E}[X^2 - 4]$
5. $\mathbb{E}[(X - 4)^2]$
6. $\text{Var}(2X - 4)$
Example 26

Solution.

1. \( \mathbb{E}[2X + 1] = 2 \times \mathbb{E}[X] + 1 = 7 \)

2. \( \mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y] = 3 - 4 = -1 \)

3. \( \mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}[X])^2 = 4 + 9 = 13 \)

4. \( \mathbb{E}[X^2 - 4] = \mathbb{E}[X^2] - 4 = 9 \)

5. \( \mathbb{E}[(X - 4)^2] = \text{Var}(X - 4) + (\mathbb{E}[X - 4])^2 = \text{Var}(X) + (\mathbb{E}[X - 4])^2 = 4 + 1 = 5 \)

6. \( \text{Var}(2X - 4) = \text{Var}(2X) = 2^2 \times \text{Var}(X) = 16 \)
Summary

In this lecture, we learned
- Expected value: definition, how to compute
- Variance: definition, how to compute