Lecture 21, 22, 23

Normal Distribution

Text: A Course in Probability by Weiss 8.5

STAT 225 Introduction to Probability Models
March 31, 2014

Agenda

1. Normal Distribution
2. Standard Normal
3. Sums of Normal Random Variables
4. Normal approximation of Binomial Distribution

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Normal Distribution

Characteristics of the Normal random variable:

Let $X$ be a Normal r.v.

- The support for $X$: $(-\infty, \infty)$
- Its parameter(s) and definition(s): $\mu$ : mean and $\sigma^2$ : variance
- The probability density function (pdf): $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $-\infty < x < \infty$
- The cumulative distribution function (cdf): No explicit form, look at the value $\Phi\left(\frac{x-\mu}{\sigma}\right)$ for $-\infty < x < \infty$ from standard normal table
- The expected value: $E[X] = \mu$
- The variance: $Var(X) = \sigma^2$

- The parameter $\mu$ determines the center of the distribution
- The parameter $\sigma^2$ determines the spread of the distribution
- Also called bell-shaped distribution
**Standard Normal** $Z \sim N(\mu = 0, \sigma^2 = 1)$

- Normal random variable $X$ with mean $\mu$ and standard deviation $\sigma$ can convert to standard normal $Z$ by the following:

$$Z = \frac{X - \mu}{\sigma}$$

- The cdf of the standard normal, denoted by $\Phi(z)$, can be found from the standard normal table.

- The probability $P(a \leq X \leq b)$ where $X \sim N(\mu, \sigma^2)$ can be compute

$$P(a \leq X \leq b) = P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

**Properties of $\Phi$**

- $\Phi(0) = .50 \Rightarrow$ Mean and Median ($50^{th}$ percentile) for standard normal are both 0

- $\Phi(-z) = 1 - \Phi(z)$

- $P(Z > z) = 1 - \Phi(z) = \Phi(-z)$

Notes

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Notes
Example 53

Let us examine $Z$. Find the following probabilities with respect to $Z$:

- $Z$ is at most $-1.75$
- $Z$ is between $-2$ and $2$ inclusive
- $Z$ is less than $0.5$

Example 53 cont'd

Solution.
Example 54

Find the following percentile with respect to $Z$

- 10th percentile
- 55th percentile
- The third quartile
- 90th percentile

Example 54 cont’d

Solution.
Example 55

Let $X$ be Normal with a mean of 20 and a variance of 49. Find the following probabilities and percentile:

1. $X$ is between 15 and 23
2. $X$ is more than 30
3. $X$ is more than 12 knowing it is less than 20
4. What is the value that is smaller than 20% of the distribution?

Example 55 cont'd

Solution.
Sums of Normal Random Variables

If $X_i$ $1 \leq i \leq n$ are independent normal random variables with mean $\mu_i$ and variance $\sigma^2_i$, respectively.

- Let $S_n = \sum_{i=1}^{n} X_i$ then $S_n \sim N(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma^2_i)$
- This can be applied for any integer $n$

Example 56

Let $X_1$, $X_2$, and $X_3$ be mutually independent, Normal random variables. Let their means and standard deviations be $3k$ and $k$ for $k = 1$, $2$, and $3$ respectively.

Find the following distributions:

- $\sum_{i=1}^{3} X_i$
- $X_1 + 2X_2 - 3X_3$
- $X_1 + 5X_3$
Example 56 cont’d

Solution.

Normal approximation of Binomial Distribution

- We can use a Normal Distribution to approximate a Binomial Distribution if $n$ is large and $p$ is close to .5
- Rule of thumb for this approximation to be valid (in this class) is $np > 5$ and $n(1-p) > 5$
- If $X \sim Binomial(n, p)$ with $np > 5$ and $n(1-p) > 5$ then we can use $X^* \sim N(\mu = np, \sigma^2 = np(1-p))$ to approximate $X$
- Notice that Binomial is a discrete distribution but normal is a continuous distribution so that $P(X^* = x) = 0 \ \forall x$
- By using continuity correction we use $P(x - 0.5 \leq X^* \leq x + 0.5)$ to approximate $P(X = x)$