Simulation of Max–Stable Processes

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Definition: Max-stable processes

A stochastic process \( \{ Y(x) \}_{x \in \mathcal{X}} \) is max-stable if there exist functions \( \{ a_n(x) \}_{x \in \mathcal{X}} > 0 \) and \( \{ b_n(x) \}_{x \in \mathcal{X}} \) such that

\[
\left\{ \frac{M_n(x) - b_n(x)}{a_n(x)} \right\}_{x \in \mathcal{X}} \overset{d}{=} \{ Y(x) \}_{x \in \mathcal{X}}, \quad n \geq 1 \tag{1}
\]

where \( M_n(x) = \max_{i=1,\ldots,n} Y_i(x) \), \( Y_1, \ldots, Y_n \) be the independent copies of \( Y \)

Remarks:

- \( \mathcal{X} \) is a finite set \( \Rightarrow \) \( Y \) is a multivariate extreme value distribution
- For any fixed \( x \in \mathcal{X} \) \( \Rightarrow \) \( Y \) is Generalized Extreme Value distribution (GEV) with parameters \( \{ \mu(x), \sigma(x), \xi(x) \} \)
Max-stable Processes

- Without loss of generality we assume each margin follow unit Fréchet distribution i.e. $P(Z \leq z) = \exp(-\frac{1}{z}) \ z > 0$
  $\Rightarrow$ this can be achieved by applying the monotone transformation
  $$Z(x) = \left[1 + \frac{\xi(x)(Y(x)-\mu(x))}{\sigma(x)}\right]^{\frac{1}{\xi(x)}} \ \forall x \in \mathcal{X}$$

- In such case, $a_n(x) = n$ and $b_n(x) = 0 \ \forall x \in \mathcal{X}$ i.e.
  $$\{\bigvee_{i=1}^n Z_i(x)\}_{x \in \mathcal{X}} \overset{d}{=} \{nZ(x)\}_{x \in \mathcal{X}}, \ n \geq 1$$
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$$\{\bigvee_{i=1}^{n} Z_i(x)\}_{x \in \mathcal{X}} \overset{d}{=} \{nZ(x)\}_{x \in \mathcal{X}}, \ n \geq 1$$
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Theorem (de Haan, 1984)

Let \( \{(\zeta_i, U_i)\}_{i \geq 1} \) be the points of a Poisson process on \((0, \infty) \times \mathbb{R}^d\) with intensity \(d\Lambda(\zeta, u) = \zeta^{-2} \, d\zeta \nu(du)\), \(\nu\) a \(\sigma\)-finite measure on \(\mathbb{R}^d\). Let \(\{Z(x)\}_{x \in \mathbb{R}^d}\) be a max-stable process with unit Fréchet margins, then there exist non-negative continuous functions \(\{f_x(y) : x, y \in \mathbb{R}^d\}\) such that

\[
\int_{\mathbb{R}^d} f_x(y) \, \nu(dy) = 1 \quad \forall x \in \mathbb{R}^d,
\]

then

\[
\{Z(x)\}_{x \in \mathbb{R}^d} \overset{d}{=} \left\{ \bigvee_{i \geq 1} \zeta_i f_x(U_i) \right\}_{x \in \mathbb{R}^d}
\]
Storm Interpretation

de Haan’s spectral representation has a storm interpretation:

\[
\{ \zeta_i \}_{i \geq 1} \rightarrow \text{storm ferocities}
\]

\[
\{ U_i \}_{i \geq 1} \rightarrow \text{storm center}
\]

\[
\zeta_i f_x(U_i) \rightarrow \text{quantity of rain at } x \text{ for the } i_{th} \text{ storm}
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Storm Interpretation cont’d

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\[ \zeta_i f_x(U_i) \rightarrow \text{quantity of rain at } x \text{ for the } i_{th} \text{ storm} \]
Theorem (Schlather, 2002)

Let \( \{ \zeta_i \}_{i \geq 1} \) be the points of Poisson process on \((0, \infty]\) with intensity \( d \Lambda(\zeta) = \zeta^{-2} d\zeta \) and \( Y_1, Y_2, \cdots \) be independent copies of a non-negative stochastic process \( \{ Y(x) \} \) such that \( \mathbb{E}[Y(x)] = 1 \) \( \forall x \in \mathbb{R}^d \). Suppose \( \{ \zeta_i \}_{i \geq 1} \) independent of \( Y_i \)'s. Then

\[
\{ Z(x) \}_{x \in \mathbb{R}^d} \overset{d}{=} \left\{ \bigvee_{i \geq 1} \zeta_i Y_i(x) \right\}_{x \in \mathbb{R}^d}
\]

is max-stable process with unit Fréchet marginals.
Schlather’s spectral representation

- There is no “storm” interpretation in Schlather’s representation.
- It allows a random shapes for the “storms” since $\{ Y(x) \}_{x \in \mathbb{R}^d}$ is a stochastic process.
Schlather’s spectral representation cont’d

- There is no “storm” interpretation in Schlather’s representation.
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![Graph showing the spectral representation](image.png)
The first max-stable model in spatial extremes literature was introduced by R. Smith by utilizing de Haan’s spectral representation:

$$\{Z(x)\}_{x \in \mathbb{R}^2} = \max_{i \geq 1} \zeta_i \phi(x - U_i; 0, \Sigma), \quad \forall x \in \mathbb{R}^2$$

where $\phi(\cdot; 0, \Sigma)$ is the bivariate normal density with mean 0 and covariance matrix $\Sigma$. 
The second model in spatial extremes literature was due to M. Schlather:

\[ \{Z(x)\}_{x \in \mathbb{R}^2} = \max_{i \geq 1} \zeta_i \sqrt{2}[Y_i(x)]_+, \quad \forall x \in \mathbb{R}^2 \]

where \( \{ Y_i(x) \}_{x \in \mathbb{R}^2} \) \( \forall i \) is the independent copies of a standard Gaussian Process with correlation function \( \rho \).
Recall the spectral representation:

\[ Z(x) = \max_{i \geq 1} \zeta_i Y_i(x), \quad \forall x \in \mathcal{X} \subset \mathbb{R}^2 \]

In order to simulate \( Z(x)_{x \in \mathcal{X}} \) we need to simulate

- Points of Poisson process with intensity \( \zeta^{-2} d\zeta \) on \((0, \infty]\)
- \( Y(x) \) at some fixed locations
- We need infinite number of these!
Recall the spectral representation:

$$Z(x) = \max_{i \geq 1} \zeta_i Y_i(x), \quad \forall x \in \mathcal{X} \subset \mathbb{R}^2$$

In order to simulate $Z(x)_{x \in \mathcal{X}}$ we need to simulate

- Points of Poisson process with intensity $\zeta^{-2} d\zeta$ on $(0, \infty]$
- $Y(x)$ at some fixed locations
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Recall the spectral representation:

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In order to simulate $Z(x)_{x \in \mathcal{X}}$ we need to simulate

- Points of Poisson process with intensity $\zeta^{-2} d\zeta$ on $(0, \infty]$
- $Y(x)$ at some fixed locations
- We need infinite number of these!
Notice that the points of Poisson process with intensity $\zeta^{-2} d\zeta$ on $(0, \infty]$ can be simulated as a strictly decreasing sequence by

$$\{\zeta_i\}_{i \geq 1} \overset{d}{=} \left\{ \frac{1}{\sum_{j=1}^{i} E_i} \right\}_{i \geq 1}, \quad E_i \overset{i.i.d.}{\sim} \text{Exp}(1)$$

It implies that if $\{Y(x)\}_{x \in \mathcal{X}}$ is uniformly bounded by $C \in (0, \infty)$ then $\zeta_i Y_i(x)$ will not contribute to $Z(x)$ for $i \geq i_0$

$\Rightarrow$ we only need to simulate $i_0$ (finite number) of these!
Simulation of Max-Stable Processes

Unconditional Simulations
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Unconditional Simulations
Goal: Given \( \{ Z(x_1) = z_1, \ldots, Z(x_n) = z_n \} \) want to get the (regular) conditional distribution of \( Z(x)_{x \in \mathcal{X}} \)

- It plays an important role for interpolating the spatial extremal surface
- If \( \{ Z(x) \}_{x \in \mathcal{X}} \) is a Gaussian process, then the conditional simulation can be thought as the interpolation of spatial mean surface
- Other than Gaussian process, the conditional distribution is difficult
Approximate \( \{Z(x)\}_{x \in \mathcal{X}} \) with a Max-Linear models

\[
\{Z(x)\}_{x \in \mathcal{X}} \approx \bigvee_{j=1}^{p} a_{i,j} Z_j := A \odot Z
\]

\( Z_j \overset{i.i.d.}{\sim} \alpha\)-Fréchet, \( a_{i,j} \) deterministic (known) such that \( a_{i,j} \geq 0 \)

Max-Linear models approximate arbitrary max-stable processes with \( p \) sufficiently large
Wang & Stoev, 2011 derive the explicit formula of conditional distribution for max-linear models.

The structure of max-linear models imposes some constraints, namely:

- \( Z_j \leq \min_{i=1, \ldots, n} \frac{z_i}{a_{i,j}} := \hat{Z}_j \), inequality constraints
- \( Z_J = \hat{Z}_J \) for some \( J \subset \{1, \ldots, p\} \), hitting constraints
A Toy Example on Max-Linear models

Suppose $n = p = 3$ and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

i.e.

$$\begin{cases} z_1 = Z_1 \\ z_2 = Z_1 \vee Z_2 \\ z_3 = Z_1 \vee Z_2 \vee Z_3 \end{cases}$$
A Toy Example cont’d

(i) If \( z = (z_1, z_2, z_3)^T = (1, 2, 3)^T \Rightarrow Z = (1, 2, 3)^T \)
The hitting constraints hold for \( Z_1, Z_2, \) and \( Z_3 \), hence \( J = (1, 2, 3) \),

(ii) If \( z = (z_1, z_2, z_3)^T = (1, 1, 3)^T \Rightarrow Z = (Z_1 = 1, Z_2 \leq 1, Z_3 = 3)^T \)
The hitting constraints hold for \( Z_1, Z_3 \), while the inequality constraint hold for \( Z_2 \) i.e. \( 0 < Z_2 \leq 1 \), hence \( J = (1, 3) \)
Algorithm

1. Compute $\hat{z}_j$, $j = 1, \cdots, p$
2. Determine all hitting scenarios $J$, and the distribution of $Z_J$ for each hitting scenarios
3. Sample the conditional distribution from the mixture distribution of each hitting scenarios

Remarks:

- To identify all hitting scenarios is NP-hard
- By the max-linear structure, with probability 1, the hitting scenarios has a nice structure which leads to an efficient algorithm
Simulations in \texttt{R}

Package \texttt{SpatialExtremes}

- \texttt{SpatialExtremes} is a \texttt{R} package written by M. Ribatet
- Statistical models for spatial extremes based on \textit{latent variables}, on \textit{copulas} and on \textit{max-stable processes} are implemented
- It also provides \textit{unconditional/conditional simulations} on max-stable processes
rmaxstab: Unconditional simulations on max-stable processes

(a) Smith Model

(b) Schlather Model
Simulation of Max-Stable Processes

Simulations in $\mathbb{R}$
condrmaxlin: Conditional sampling for Max-Linear models

Figure: Four conditional simulations for Smith model, $p = 10000$. 
Thanks! Questions?
References

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