Modeling Spatial Extremes, Part III

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Spatial Extremes

- Suppose we observe the maximum temperature of a certain long period at spatial location \( s \), denoted by \( X(s) \).
  - \( X(s) \) has a GEV distribution.
- What about the joint distribution of \( X(s_1), \ldots, X(s_n) \) for any locations of \( s_1, \ldots, s_n \)?
- Need to define a stochastic process with GEV marginal distributions.
Let
\begin{itemize}
\item \{(\xi_i, s_i), i \geq 1\} be point process on \((0, \infty) \times S\) with intensity measure 
\[\xi^{-2}d\xi \times \nu(ds)\], where \(S\) is a spatial domain in \(R^d\), and \(\nu\) a positive
measure on \(S\), and
\[f(s, t) \geq 0, \int_S f(s, t)\nu(ds) = 1, \forall t \in R^d.\]
\item \(Y(t) = \max_{i \geq 1}\{\xi_i f(s_i, t)\}, t \in R^d.\)
\item Then for any subset \(T\) of \(R^d\),
\[P\{Y(t) \leq y_t \text{ for all } t \in T\} = \exp \left[- \int_S \max_{t \in T} \left(\frac{f(s, t)}{y_t}\right) \nu(ds)\right].\]
\end{itemize}
One Construction (Continued)

- For $T = \{t\}$, $P(Y(t) \leq y_t) = \exp[-1/y_t]$.
- $Y(t)$ has the Fréchet marginal distribution.
- Finite dimensional distribution:

$$P(Y(t_1) \leq y_1, \ldots, Y(t_n) \leq y_n) = \exp\left[ - \int S \max_{1 \leq i \leq n} \left( \frac{f(s, t_i)}{y_t} \right) \nu(ds) \right].$$

- Defined a process with a Fréchet marginal distribution.
  Transformation to yield any marginal GEV distribution.
Gaussian extreme value process (Smith, 1990): Consider $S = T = R^d$ and $f(s, t)$ to be the density function of $N(s - t, \Sigma)$, i.e.,

$$f(s, t) = (2\pi)^{-d/2} |\Sigma|^{-1} \exp \left\{ -\frac{1}{2} (s - t)' \Sigma^{-1} (s - t) \right\}.$$
If \( \nu \) is degenerate and has its mass at points 1, 2, \ldots, \( L \), then

\[
\int f(s, t) \nu(ds) = \sum_{l=1}^{L} f(l, t) = 1
\]

and

\[
P\{ Y(t) \leq y_t \text{ for all } t \in T \} = \exp \left[ - \sum_{l=1}^{L} \max_{t \in T} \left( \frac{f(l, t)}{y_t} \right) \right].
\]
We have constructed a particular class of spatial extremal process. In general, what kind of stochastic processes is appropriate for modelling spatial extremes? The answer is spatial max-stable processes.

GEV is max-stable in the sense that if \( Z \sim GEV(\xi, \mu, \sigma) \), then for \( Z, Z_1, \ldots, Z_n \) i.i.d, there exist \( b_n \) and \( a_n > 0 \) such that

\[
\frac{\max_i Z_i - b_n}{a_n}
\]

has the same distribution as \( Z \).
Max-stable process

Let \( \{ Y_i(s), s \in R^d \}, i = 1, \ldots, n \) be i.i.d stochastic processes. If there are functions \( \mu_n(s), \sigma_n(s) \) such that the process

\[
\left\{ \max_{i=1,\ldots,n} \frac{Y_i(s) - \mu_n(s)}{\sigma_n(s)}, s \in R^d \right\} \to \{ Z(s), s \in R^d \},
\]

then \( Z(s) \) is max-stable in the sense that there exist \( b_n(s) \) and \( a_n(s) > 0 \) such that the process

\[
\frac{\max_i Z_i(s) - b_n(s)}{a_n(s)}
\]

has the same distribution as \( Z(s) \).

Max-stable process has a GEV marginal distribution.
A max-stable process $Z(s)$ for $s \in R^d$ can be expressed as

$$Z(s) = \max_i \xi_i f(s - T_i)$$

where $(\xi_i, T_i)$ are points of a Poisson process with intensity measure $u^{-2} du \times \rho(dt)$ where $\rho$ is a finite measure and $f$ is a non-negative function.
Let $\xi_i, i \geq 1$ be the points of a Poisson process on $(0, \infty]$ with intensity $\xi^{-2}d\xi$ and $Y_1(s), Y_2(s), \ldots$ be i.i.d. replications of a stochastic process $Y(s)$ such that $E\max\{0, Y(s)\} = 1$, for all $s \in R^d$, where the processes $Y_i(s)$ and the Poisson process are assumed to be independent. Then

$Z(x) = \max_{i \geq 1} \xi_i \max\{0, Y_i(x)\}$ is a max-stable process with Fréchet margins.

Different choices of $Y(\cdot)$ yield different max-stable processes.

\[
P(Z(s_1) \leq z_1, \ldots, Z(s_n) \leq z_n) = \exp \left( - E \left[ \max_{i=1,\ldots,n} \left( \frac{Y_+(s_i)}{z_i} \right) \right] \right)
\]

where $Y_+(s) = \max\{0, Y(s)\}$. 

Extremal coefficient

Recall

\[ P(Z(s_1) \leq z_1, \ldots, Z(s_n) \leq z_n) = \exp \left( -E \left[ \max_{i=1, \ldots, n} \left( \frac{Y_+(s_i)}{z_i} \right) \right] \right) \]

Hence

\[ P(Z(s_1) \leq z, \ldots, Z(s_n) \leq z) = \exp \left( -\frac{1}{z} E \left[ \max_{i=1, \ldots, n} Y_+(s_i) \right] \right) \]
\[ = \{ \exp(-1/z) \}^{\eta(s_1, \ldots, s_n)} \]
\[ = P(Z(s) \leq z)^{\eta(s_1, \ldots, s_n)} \]

\[ \eta(s_1, \ldots, s_n) = E \left[ \max_{i=1, \ldots, n} Y_+(s_i) \right]. \]

- \( \eta(s_1, \ldots, s_n) \geq 1 \) is called the extremal coefficient.
- \( \eta(s_1, \ldots, s_n) = n \) implies independence.
Consider a pair \((x, x + h)\). Due to stationarity

\[ \eta(x, x + h) = E[\max(Y_+(x), Y_+(x + h))] = \theta(h). \]

\[ P(Z(x) \leq z, Z(x + h) \leq z) = [P(Z(x) \leq z)P(Z(x + h) \leq z)]^{\theta(h)/2}. \]

\( \theta(h) \in [1, 2] \) and \( \theta(h) = 2 \) implies independence.
Some special models

- **Brown-Resnick (1987)** \( Y_i(s) = \exp\{W_i(s) - \gamma(s)\} \), \( W_i(s) \) intrinsically stationary Gaussian with a semi-variogram \( \gamma \).

  \[ \theta(h) = 2\Phi(\sqrt{\gamma(h)/2}). \]

- **Smith (1990)** \( Y_i(s) = f(s - s_i), \{s_i, i = 1, 2, \ldots\} \) a homogeneous Poisson point process on \( R^d \).

  \[ \theta(h) = 2\Phi\left(\frac{\sqrt{h'\Sigma^{-1}h}}{2}\right). \]

- **Schlather (2002)** \( Y_i(s) = \sqrt{2\pi}e_i(s) \), where \( e_i(s) \) standard Gaussian process.

  \[ \theta(h) = 1 + \sqrt{(1 - \rho(h))/2}. \]
Pairwise likelihood function

Recall:

\[ P(Z(s_1) \leq z_1, \ldots, Z(s_n) \leq z_n) = \exp \left( -E \left[ \max_{i=1,\ldots,n} \left( \frac{Y_+(s_i)}{z_i} \right) \right] \right). \]

- Pairwise density function: For any pair of locations \((s_1, s_2)\), write

\[ v(z_1, z_2) = E \left[ \max_{i=1,2} \left( \frac{Y_+(s_i)}{z_i} \right) \right]. \]

The joint density of \(Z(s_1)\) and \(Z(s_2)\) is

\[ f(z_1, z_2) = \left( \frac{\partial v}{\partial z_1} \frac{\partial v}{\partial z_2} - \frac{\partial^2 v}{\partial z_1 \partial z_2} \right) \exp(-v(z_1, z_2)). \]

- Pairwise likelihood and composite likelihood to estimate parameters in the \(Y\) process.