Estimation methods for Levy based models of asset prices

José Enrique Figueroa-López

Financial mathematics seminar

Department of Statistics and Applied Probability

UCSB

September 2006

Objectives

To *discuss* some aspects of **Lévy processes** related to

1. *Stochastic modeling of historical asset prices*
2. *Statistical inference: Nonparametric methods*

Program

I. Models of historical prices

II. Lévy processes

III. Standard statistical methods

IV. Recent nonparametric methods based on high-frequency sampling

Modeling of historical asset prices

Problem: "Construct" stochastic processes that account for the known features of stock prices dynamics.

Motivations: Sensible allocation of money in a portfolio of assets. Risk assessment.

What has been done?

- Geometric Brownian Motion
- Lévy based modeling
- Stochastic volatility models

Geometric Brownian Motion

- The model: *The "return" (per dollar invested) of the stock during a small time span dt is approx. normally distributed with constant mean and variance*:

$$\frac{S_{t+dt} - S_t}{S_t} \approx \underbrace{\mu}_{\text{Mean return}} dt + \underbrace{\sigma}_{\text{Volatility}} d \underbrace{W_t}_{\text{B.M.}} .$$

More precisely,

$$\underbrace{\log \frac{S_{t+\Delta t}}{S_t}}_{\text{Log Return on } [t, t+\Delta t)} = (\mu - \frac{\sigma^2}{2})\Delta t + \sigma \underbrace{(W_{t+\Delta t} - W_t)}_{\sim N(0, \Delta t)} .$$

- Partial theoretical justification: The central limit theorem.

- What is a Standard Brownian Motion?

  A random quantity W_t evolving in time $t \geq 0$ in such a way that

  - The change (*increment*) $W_{t+\Delta t} - W_t$ during a period $[t, t + \Delta t]$

    (1) is independent of the "past" $\{W_s : s \leq t\}$ and

    (2) has *Normal distribution* law depending only on the time span Δt

  - The process is continuous (no jumps or "sudden" changes) and $W_0 = 0$.

- Implications:

  - *Efficiency*: Future prices depends on the past only through the present value (Markov property)
  - Log returns in disjoint periods are independent
  - Continuously varying stock prices

- Empirical evidence:
  - The distribution of “short-term” returns exhibit *heavy tails* and *high kurtosis*.
  - Sudden changes in prices due to arrival of information
- Natural questions:
  - What is wrong: Independence and/or stationary of increments, continuity? Probably all of them!! Which are more implausible?
  - Can we construct a model that allows fat-tail marginal distributions, while preserving the statistical qualities of the increments and continuity? No!!

Jump-based modeling

Does it make sense?

- The prices moves discontinuously driven by discrete trades
- "Sudden large" changes due to arrival of information

Lévy Model: A random quantity X_t evolving in time $t \geq 0$ in such a way that

- The change $X_{t+\Delta t} - X_t$ during the period $[t, t+\Delta t]$
  (1) is independent of the "past" $\{X_s : s \leq t\}$ and
  (2) has distribution depending only on the time span Δt
- The process could exhibit sudden changes (jumps), but these occur at unpredictable times (no fixed jump times). Also, $X_0 = 0$.

Message: Lévy random processes $\{X_t\}_{t \geq 0}$ (if they exist?) are the most natural extensions of Brownian motion.

Geometric Lévy Motion

- The Model:

$$\underbrace{\log \frac{S_{t+\Delta t}}{S_t}}_{\text{Log Return on } [t, t+\Delta t)} = \underbrace{X_{t+\Delta t} - X_t}_{\text{Increment of a Levy Process}} \iff S_t = S_0 e^{X_t}$$

- Implications:

1. Equally-spaced Log returns

$$R_i := \log \frac{S_{i\Delta t}}{S_{(i-1)\Delta t}} = X_{i\Delta t} - X_{(i-1)\Delta t},$$

are independent and identically distributed with law $\mathcal{L}(X_{\Delta t})$.

2. $\mathbb{E}X_t = mt$ and $\mathrm{Var}X_t = \sigma^2 t$.

Pitfalls of Geometric Lévy models

Empirical evidence: [Cont: 2001]

- Volatility clustering: High-volatility events tend to cluster in time
- Leverage phenomenon: volatility is negatively correlated with returns
- Some sort of long-range memory: Returns do not exhibit significant autocorrelation; however, the autocorrelation of *absolute returns* decays slowly as a function of the time lag.

Conclusion: “Need” for increasingly more complex models

Other issues:

- Measurement of volatility?
- Measurement of dependence or correlation?

Other Lévy-based alternatives

Time-changed Lévy process: [Carr, Madan, Geman, Yor etc.]

$$\log S_t/S_0 = X_{T_t},$$

T_t is an increasing random process (Random Clock).

Stochastic volatility driven by Lévy processes: [B-N and Shephard]

$$\log S_t/S_0 = \int_0^t (\mu - \frac{\sigma_t^2}{2})dt + \int_0^t \sigma_t dW_t.$$
$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dX_{\lambda t},$$

$\{X_t\}_{t\geq 0}$ is a Lévy process that is nondecreasing.

Stochastic volatility with jumps in the return:

$$\log S_t/S_0 = \int_0^t \mu_u du + \int_0^t \sigma_u dW_u + \begin{cases} X_t \\ \sum_{u \le t} h(\Delta X_u, u) \end{cases},$$

$\Delta X_t =$ Size of the jump of X at time t, and $h(0,\cdot) = 0$.

SDE with jumps in the returns and the volatility: [Todorov 2005]

$$\log S_t/S_0 = \mu t + + \int_0^t \sigma_{u-} dW_u + \sum_{u \le t} h(\Delta X_u),$$

$$\sigma_t^2 = \sum_{u \le t} f(t-u) k(\Delta X_u),$$

$h(0) = k(0) = 0$

Summary

1. Exponential Lévy models are some of the simplest and most practical alternatives to the shortfalls of the geometric Brownian motion.

2. Capture several stylized empirical features of historical returns.

3. Limitations: Lack of stochastic volatility, leverage, quasi-long-memory, etc.

4. Lévy processes have been increasingly becoming an important tool in asset price modeling.

Statistical properties of Lévy processes

Characterization and parameters

- The statistical law of the process is determined by the distribution of X_t
- Three parameters, two reals σ^2, b, and a measure $\nu(dx)$, so that

$$(1) \quad X_t = bt + \sigma W_t + \lim_{\varepsilon \downarrow 0} \left\{ X_t^{\varepsilon} - t m_{\varepsilon} \right\},$$

$$(2) \quad X^{\varepsilon} = \text{Compound Poisson Process with intensity } m_{\varepsilon}$$

$$(3) \quad N_t([a,b]) := \sum_{u \leq t} \mathbf{1}[\Delta X_u \in [a,b]] \sim \text{Poisson}\left(t\nu([a,b])\right),$$

- **Lévy Density**: A nonnegative function s such that

$$\nu([a,b]) = \int_a^b s(x) dx,$$

Intuition: $s(x)$ dictates the frequency of jumps with sizes near to x

Important remarks

Necessary and sufficient conditions to be a Lévy density:

$$\int_{-1}^{1} s(x)x^2 dx < \infty \quad \text{and} \quad \int_{|x|>1} s(x)dx < \infty.$$

Consequence:

- A Lévy density is *not* a probability density function:

$$\int_{-\infty}^{\infty} s(x)dx = \infty \Rightarrow s(x) \overset{|x|\to 0}{\longrightarrow} \infty \iff$$ inf. many jumps of arbitrarily small size

- It is easier to specify a Lévy process via the Lévy density than via the marginal distribution of X_1. Thus, it is easier to model the jump behavior of the process.