Estimation methods for Levy based models of asset prices

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Objectives

To discuss some aspects of Lévy processes related to

- 1. Stochastic modeling of historical asset prices
- 2. Statistical inference: Nonparametric methods

Program

- I. Models of historical prices
- II. Lévy processes
- **III.** Standard statistical methods
- **IV.** Recent nonparametric methods based on high-frequency sampling

Modeling of historical asset prices

Problem: "Construct" stochastic processes that account for the known features of stock prices dynamics.

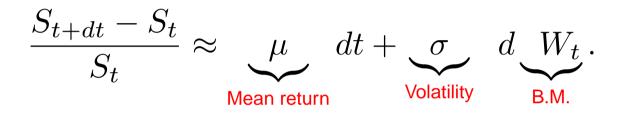
Motivations: Sensible allocation of money in a portfolio of assets. Risk assessment.

What has been done?

- Geometric Brownian Motion
- Lévy based modeling
- Stochastic volatility models

Geometric Brownian Motion

• The model: The "return" (per dollar invested) of the stock during a small time span dt is approx. normally distributed with constant mean and variance:



More precisely,

$$\underbrace{\log \frac{S_{t+\Delta t}}{S_t}}_{\text{Log Return on }[t,t+\Delta t)} = (\mu - \frac{\sigma^2}{2})\Delta t + \sigma \underbrace{(W_{t+\Delta t} - W_t)}_{\sim N(0,\Delta t)}.$$

• Partial theoretical justification: The central limit theorem.

• What is a Standard Brownian Motion?

A random quantity W_t evolving in time $t \ge 0$ in such a way that

- The change (*increment*) $W_{t+\Delta t} - W_t$ during a period $[t, t + \Delta t]$ (1) is independent of the "past" $\{W_s : s \leq t\}$ and

(2) has Normal distribution law depending only on the time span Δt

- The process is continuous (no jumps or "sudden" changes) and $W_0 = 0.$
- Implications:
 - *Efficiency*: Future prices depends on the past only through the present value (Markov property)
 - Log returns in disjoint periods are independent
 - Continuously varying stock prices

- Empirical evidence:
 - The distribution of "short-term" returns exhibit *heavy tails* and *high kurtosis*.
 - Sudden changes in prices due to arrival of information
- Natural questions:
 - What is wrong: Independence and/or stationary of increments, continuity? Probably all of them!! Which are more implausible?
 - Can we construct a model that allows fat-tail marginal distributions, while preserving the statistical qualities of the increments and continuity? No!!

Jump-based modeling

Does it make sense?

- The prices moves discontinuously driven by discrete trades
- "Sudden large" changes due to arrival of information

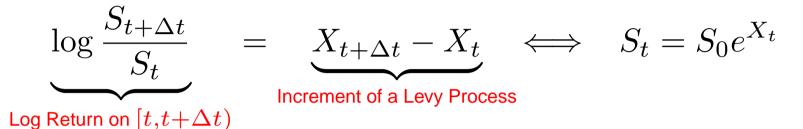
Lévy Model: A random quantity X_t evolving in time $t \ge 0$ in such a way that

- The change $X_{t+\Delta t} X_t$ during the period $[t, t + \Delta t]$ (1) is independent of the "past" $\{X_s : s \leq t\}$ and (2) has distribution depending only on the time span Δt
- The process could exhibit sudden changes (jumps), but these occur at unpredictable times (no fixed jump times). Also, $X_0 = 0$.

Message: Lévy random processes $\{X_t\}_{t\geq 0}$ (if they exist?) are the most natural extensions of Brownian motion.

Geometric Lévy Motion

• The Model:



- Implications:
 - 1. Equally-spaced Log returns

$$R_i := \log \frac{S_{i\Delta t}}{S_{(i-1)\Delta t}} = X_{i\Delta t} - X_{(i-1)\Delta t},$$

are independent and identically distributed with law $\mathcal{L}(X_{\Delta t})$.

2.
$$\mathbb{E}X_t = mt$$
 and $\operatorname{Var}X_t = \sigma^2 t$.

Pitfalls of Geometric Lévy models

Empirical evidence: [Cont: 2001]

- Volatility clustering: High-volatility events tend to cluster in time
- Leverage phenomenon: volatility is negatively correlated with returns
- Some sort of long-range memory: Returns do not exhibit significant autocorrelation; however, the autocorrelation of *absolute returns* decays slowly as a function of the time lag.

Conclusion: "Need" for increasingly more complex models

Other issues:

- Measurement of volatility?
- Measurement of dependence or correlation?

Other Lévy-based alternatives

Time-changed Lévy process: [Carr, Madan, Geman, Yor etc.]

$$\log S_t / S_0 = X_{T_t},$$

 T_t is an increasing random process (Random Clock).

Stochastic volatility driven by Lévy processes: [B-N and Shephard]

$$\log S_t / S_0 = \int_0^t (\mu - \frac{\sigma_t^2}{2}) dt + \int_0^t \sigma_t dW_t d\sigma_t^2 = -\lambda \sigma_t^2 dt + dX_{\lambda t},$$

 $\{X_t\}_{t\geq 0}$ is a Lévy process that is nondecreasing.

Stochastic volatility with jumps in the return:

$$\log S_t/S_0 = \int_0^t \mu_u du + \int_0^t \sigma_u dW_u + \begin{cases} X_t \\ \sum_{u \le t} h(\Delta X_u, u) \end{cases},$$

 $\Delta X_t =$ Size of the jump of X at time t, and $h(0, \cdot) = 0$.

SDE with jumps in the returns and the volatility: [Todorov 2005]

$$\log S_t / S_0 = \mu t + \int_0^t \sigma_{u^-} dW_u + \sum_{u \le t} h(\Delta X_u),$$
$$\sigma_t^2 = \sum_{u \le t} f(t - u) k(\Delta X_u),$$
$$h(0) = k(0) = 0$$

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Summary

- 1. Exponential Lévy models are some of the simplest and most practical alternatives to the shortfalls of the geometric Brownian motion.
- 2. Capture several stylized empirical features of historical returns.
- 3. Limitations: Lack of stochastic volatility, leverage, quasi-long-memory, etc.
- 4. Lévy processes have been increasingly becoming an important tool in asset price modeling.

Statistical properties of Lévy processes

Characterization and parameters

- The statistical law of the process is determined by the distribution of X_t
- Three parameters, two reals σ^2 , b, and a measure u(dx), so that

(1)
$$X_t = bt + \sigma W_t + \lim_{\varepsilon \downarrow 0} \{X_t^\varepsilon - tm_\varepsilon\},$$

(2) $X^{\varepsilon} =$ Compound Poisson Process with intensity m_{ε}

- (3) $N_t([a,b]) := \sum_{u \le t} \mathbf{1}[\Delta X_u \in [a,b]] \sim \text{Poisson}(t\nu([a,b]),$
- Lévy Density: A nonnegative function s such that

$$\nu([a,b]) = \int_a^b \frac{s(x)dx}{dx},$$

Intuition: s(x) dictates the frequency of jumps with sizes near to x

Important remarks

Necessary and sufficient conditions to be a Lévy density:

$$\int_{-1}^{1} s(x)x^{2}dx < \infty \text{ and } \int_{|x|>1} s(x)dx < \infty.$$

Consequence:

• A Lévy density is *not* a probability density function:

$$\int_{-\infty}^{\infty} s(x) dx = \infty \Rightarrow s(x) \xrightarrow{|x| \to 0} \infty \iff$$
 inf. many jumps of arbitrarily small size

 It is easier to specify a Lévy process via the Lévy density than via the marginal distribution of X₁. Thus, it is easier to model the jump behavior of the process.