Brief tutorial of Lévy processes

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Abstract

Some fundamental properties related to Lévy processes are discussed. Topics include infinitely divisible distributions, Lévy-Khintchine formula, Poisson random measures, Lévy-Itô decomposition, series representations, and density transformations.

1 Basic properties

- In short, a **Lévy process** $X = \{X_t\}_{t\geq 1}$ is a $\mathbb{R}^d$-valued process with **independent and stationary increments** whose paths are right-continuous with left-limits and that has **no fixed jump-times**.
- The law of X is uniquely determined by the distribution of X_1 (*Why?*).
- Clearly, the marginal distributions are **infinitely-divisible**:
  - ξ is said to be infinitely divisible if for any n, one can construct n iid r.v. $\xi_1, \dots, \xi_n$ such that
    $$\xi \overset{\mathfrak{D}}{=} \xi_1 + \cdots + \xi_n.$$
  - It has a close relation to limits in distribution of a array of row-wise i.i.d. r.v.:
    - ξ is infinitely divisible iff there exists $\{\xi_{n,k}\}_{k=1}^{k_n}$, i.i.d. for each n, such that
      $$\sum_{k=1}^{k_n} \xi_{n,k} \xrightarrow{\mathfrak{D}} \xi, \quad \text{as} \quad n \to \infty.$$
  - It has an explicit characteristic function:
    - ξ is infinitely divisible iff
      $$\mathbb{E}e^{iu\xi} = \exp\left\{-\frac{1}{2}\langle u, Au\rangle + i\langle b, u\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle \mathbf{1}_{|x|\leq 1}\right)\nu(dx)\right\},$$

for some symmetric nonnegative-definite matrix A, a vector $b \in \mathbb{R}^d$, and a measure ν on $\mathbb{R}^d$ such that

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int (|x|^2 \wedge 1)\nu(dx) < \infty.$$

- Some small-time asymptotic characterizations:

  1. $\frac{1}{\sqrt{t}} X_t \xrightarrow{\mathfrak{D}} Z$, where $Z := (Z^1, \ldots, Z^d)$ is a multivariate Gaussian vector with variance-covariance function A.
  2. For any $g : \mathbb{R}^d \to \mathbb{R}$ that is (ν-a.e.) continuous, bounded, and vanishes in a neighborhood of the origin:

  $$\lim_{t \to 0} \frac{1}{t} \, \mathbb{E}\, g(X_t) = \int g(x)\nu(dx).$$

- An application: consistent estimation.

  - Suppose that we sample X at m equally spaced point with span $1/n$: we observe $X_{\frac{k}{n}}$, for $k = 1, \ldots, m$.
  - Consider the statistics:

  $$\beta_{n,m}(g) := \frac{n}{m} \sum_{k=1}^{m} g\left(X_{\frac{k}{n}} - X_{\frac{k-1}{n}}\right).$$

  Then,

  $$\lim_{n\to\infty} \lim_{m \to \infty} \beta_{n,m}(g) = \int g(x)\nu(dx),$$

  almost surely (*Why?*). Actually, it can be proved that

  $\beta_{n,m_n}(g) \to \int g(x)\nu(dx)$ (in probability) whenever

  $$\begin{aligned} 1/n &= \text{Time span between observations} \to 0 \\ m_n/n &= \text{Sampling time horizon} \to \infty. \end{aligned}$$

2 Poisson random measures

2.1 Basic definition and properties

- Let $\mathcal{S}$ be a Borel subset of $\mathbb{R}^d$ and m be a Borel measure on $\mathcal{S}$ that is σ-finite. A collection $\{N(B) : B \in \mathcal{S}\}$ of $\bar{\mathbb{Z}}_+$-valued random variables is called a **Poisson random measure** (or process) on S with mean measure m if

  (1) for every B, $N(B)$ is a Poisson random variable with mean $m(B)$;

  (2) if $B_1, \ldots, B_n$ are disjoint, then $N(B_1), \ldots, N(B_1)$ are independent;

  (3) for every ω, $N(\cdot;\omega)$ is a measure on $\mathcal{S}$.

- It can be proved that (a.s.), $N(\cdot;\omega)$ is an atomic measure; that is to say, there exists countably many (random) points $\{\mathbf{x}_i\}_i \subset \mathcal{S}$ such that

  $$N(B) = \#\{i : \mathbf{x}_i \in B\}.$$

  Equivalently,

  $$N(\cdot) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i}(\cdot).$$

  Sometimes, the random points $\{\mathbf{x}_i\}_i$ is called a **Poisson point process** on the $\mathcal{S}$ with mean measure m.

2.2 Construction by conditioning

- **Key property:** Suppose that $m(B) < \infty$. Then, conditioning to $N(B) = k$, the points of $\{\mathbf{x}_i\}$ in B, denote by $\{\mathbf{x}_i^B\}$ are distributed according to the distribution $m(\cdot)/m(B)$, independently from each other:

  $$\mathbb{P}\{\mathbf{x}_i^B \in C | N(B) = k\} = m(C)/m(B).$$

- Construction procedure:

  1. Suppose that $B_1, B_2, \ldots$ is a partition of $\mathcal{S}$ such that $m(B_j) < \infty$.
  2. Generate $n_j \sim Poiss(m(B_j))$.
  3. Independently, generate n_j-points, say $\{\mathbf{x}_i^j\}_{i=1}^{n_j}$, according to the distribution $m(\cdot)/m(B_j)$.
  4. Define
     $$N(B) = \#\{(i,j) : \mathbf{x}_i^j \in B\}.$$

- Example:

  - Take $\mathbb{R}_+$ and $m(dt) = dt$. Then, if $\Gamma_1 < \Gamma_2 < \ldots$ denotes the point process associated with a Poisson random measure on $\mathbb{R}_+$ with mean measure m, then $\Gamma_i - \Gamma_{i-1}$ are independent with common $exp(1)$ distribution (*Why?*). Such a Poisson process is said to be homogeneous and the sequence $\{\Gamma_i\}_i$ is also called the arrival times of the Poisson process.

2.3 Transformations of Poisson point processes

- Suppose that $T : \mathcal{S} \to \mathcal{S}' \subset \mathbb{R}^{d'}$ be measurable and let

  $$N(\cdot) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i}(\cdot),$$

be a Poisson random measure with mean measure m. Then, the random measure associated with the transformed points $\mathbf{x}_i' := T(\mathbf{x}_i)$, namely

$$N'(\cdot) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i'}(\cdot).$$

is also a Poisson random measure with mean measure

$$N'(B) := N(\{\mathbf{x} : T(\mathbf{x}) \in B\}).$$

- Suppose that $\sigma(\mathbf{x}, d\mathbf{x}')$ is a probability measure on $\mathcal{S}' \subset \mathbb{R}^{d'}$, for each $\mathbf{x} \in \mathcal{S}$, and that

  $$N(\cdot) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i}(\cdot),$$

  is a Poisson random measure with mean measure m. For each i, generate a r.v. $\mathbf{x}_i'$ according $\sigma(\mathbf{x}_i, d\mathbf{x}')$ (independently from any other variable). Then,

  $$N'(\cdot) = \sum_{i=1}^{\infty} \delta_{(\mathbf{x}_i, \mathbf{x}_i')}(\cdot),$$

  is a Poisson random measure with mean measure

  $$m'(d\mathbf{x}, d\mathbf{x}') = \sigma(\mathbf{x}, d\mathbf{x}')m(d\mathbf{x}).$$

2.4 Integration with respect to a Poisson random measure

- Since $N(\cdot;\omega)$ is an atomic random measure for each ω, say $N(\cdot) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i}(\cdot)$, one can define the integral

  $$\int_{\mathcal{S}} f(\mathbf{x})N(d\mathbf{x}) = \sum_{i=1}^{\infty} f(\mathbf{x}_i),$$

  for any measurable **nonnegative** deterministic function f. This is a $\bar{\mathbb{R}}_+$-valued r.v.

- Moreover,

  $$\mathbb{E}\left[e^{-\int f(\mathbf{x})N(d\mathbf{x})}\right] = \exp\left\{-\int \left(1 - e^{-f(\mathbf{x})}\right) m(d\mathbf{x})\right\},$$

  $$\mathbb{E}\left[\int f(\mathbf{x})N(d\mathbf{x})\right] = \int f(\mathbf{x})m(d\mathbf{x}).$$

- Notice that if $B \subset \mathcal{S}$ is such that $m(B) < \infty$, then

  $$\int_B f(\mathbf{x})N(d\mathbf{x}) := \sum_{i:\mathbf{x}_i \in B} f(\mathbf{x}_i),$$

  is a well-defined $\mathbb{R}^d-$valued r.v. for any measurable function $f : \mathcal{S} \to \mathbb{R}^d$. Its characteristic function is

  $$\mathbb{E}\left[e^{\mathbf{i}\left\langle \int_B f(\mathbf{x})N(d\mathbf{x}), \mathbf{u}\right\rangle}\right] = \exp\left\{-\int_B \left(e^{\mathbf{i}\langle f(\mathbf{x}), \mathbf{u}\rangle} - 1\right) m(d\mathbf{x})\right\}.$$

- If $B_1, \dots, B_m$ are disjoints sets in $\mathcal{S}$ with finite measure, then

  $$\int_{B_1} f(\mathbf{x})N(d\mathbf{x}), \dots, \int_{B_m} f(\mathbf{x})N(d\mathbf{x}).$$

  are independent (*Why? Hint: Consider linear combinations.*).

3 Lévy-Itô Representation

Theorem 1 ***[13.4, Kallenberg]*** *Let $\{X(t)\}_{t\geq 0}$ be an rcll process in $\mathbb{R}^d$ with $X(0) = 0$. Then, X has independent increments without fixed jumps times if and only if, a.s.*

$$\mathbf{X}(t) = \mathbf{b}_t + \mathbf{G}_t + \int_0^t \int_{\|\mathbf{x}\|\leq 1} \mathbf{x}\,(N-m)(ds, d\mathbf{x}) + \int_0^t \int_{\|\mathbf{x}\|>1} \mathbf{x}\,N(ds, d\mathbf{x}), \forall t \geq 0, \tag{1}$$

for some continuous function $\mathbf{b}$ with $\mathbf{b}_0 = 0$, some continuous centered Gaussian process $\mathbf{G}$ with independent increments and $\mathbf{G}_0 = 0$, and some independent Poisson random measure N on $[0,\infty) \times \mathbb{R}_0^d$ with mean measure m satisfying

$$\int_0^t \int_{\mathbb{R}_0^d} (\mathbf{x}^2 \wedge 1)\, m(ds, d\mathbf{x}) < \infty, \quad t > 0. \tag{2}$$

The representation is almost surely unique, and all functions $\mathbf{b}$, processes G, and measures m with the stated properties may occur.

Remark 1 ***[19.2, Sato]*** *In the equation (1), the second term is understood as the limit*

$$\int_0^t \int_{\|\mathbf{x}\|\leq 1} \mathbf{x}(N-m)(ds, d\mathbf{x}) = \lim_{\varepsilon\downarrow 0} \int_0^t \int_{\varepsilon<\|\mathbf{x}\|\leq 1} \mathbf{x}\,(N-m)(ds, d\mathbf{x}). \tag{3}$$

where the limit exists almost surely. Furthermore, the convergence is uniform in t on any bounded interval.

4 Simulation of Lévy processes

4.1 Brief overview

Accurate path simulation of a pure jump Lévy processes $\mathbf{X} = \{\mathbf{X}(t)\}_{t\in[0,T]}$, regardless of the relatively simple statistical structure of their increments, present some challenging problems when dealing with *infinite activity* (namely, processes with infinite Lévy measure). Just try to conceive that in this case the jump times are in fact dense on $[0,\infty)$ (see Theorem 21.3 of [4]).

One of the most popular simulation schemes is based on the generation of *discrete skeletons*. Namely, the discrete skeleton of $\mathbf{X}$ based on equally spaced time points is defined by

$$\widetilde{\mathbf{X}}(t) = \sum_{k=1}^{\infty} \mathbf{X}(k-1/n)\, \mathbf{1}(t \in [(k-1)/n, k/n)) = \sum_{k=1}^{\infty} \Delta_k \mathbf{1}(t \geq k/n),$$

where $\Delta_k = \mathbf{X}(k/n) - \mathbf{X}(k-1/n)$ are i.i.d. with common distribution $\mathcal{L}(\mathbf{X}(1/n))$. The main drawback to the previous scheme is the fact that most often a r.v. with distribution $\mathcal{L}(\mathbf{X}(1/n))$ is not easily generated.

The second easiest scheme would be to approximate the Lévy process by finite activity Lévy processes. That is, the Lévy-Itô decomposition of sample paths establishes that a.s. the process

$$\mathbf{X}_\varepsilon(t) \equiv t\left(\mathbf{b} - \int_{\|\mathbf{x}\|\geq\varepsilon} \mathbf{x}\,\nu(d\mathbf{x})\right) + \sum_{s\leq t} \Delta\mathbf{X}(s)\mathbf{1}(\|\Delta\mathbf{X}(s)\| \geq \varepsilon) \tag{4}$$

converges uniformly on any bounded interval, and a.s. the limiting process coincides with the paths of $\mathbf{X}$ (above, $\Delta\mathbf{X}(t) \equiv \mathbf{X}(t) - \mathbf{X}(t^-)$). The process $\sum_{s\leq t} \Delta\mathbf{X}(s)\mathbf{1}(\|\Delta\mathbf{X}(s)\| \geq \varepsilon)$ can be simulated using a *compound Poisson process* of the form $\sum_{k=1}^{N_t^\varepsilon} Y_k^\varepsilon$, where N_t^ε is a homogeneous Poisson process with intensity $\nu(\|\mathbf{x}\| \geq \varepsilon)$ and where $\{Y_k^\varepsilon\}_{k=1}^\infty$ are i.i.d with common distribution

$$\nu_\varepsilon(d\mathbf{x}) \equiv \mathbf{1}(\|\mathbf{x}\| \geq \varepsilon)\frac{\nu(d\mathbf{x})}{\nu(\|\mathbf{x}\| \geq \varepsilon)}.$$

Clearly, such a scheme is unsatisfactory because all jumps smaller than ε are totally ignored. An alternative method of simulation is based on time series representations of the form

$$\mathbf{X}(t) = \mathbf{b}t + \sum_{i=1}^{\infty} \left[\mathbf{H}\left(\Gamma_i, \mathbf{V}_n\right)\mathbf{1}(U_i \leq t) - t\mathbf{c}_i\right],$$

which will be explained next.

4.2 Simulations based on series representations

Throughout, $\mathbf{X} = \{\mathbf{X}(t)\}_{t\in[0,T]}$ is a Lévy process on $\mathbb{R}^d$ with Lévy measure ν and without Gaussian part (which can be separately simulated). The results below are presented in [3]. A series representation for $\mathbf{X}$ can be derived from a series representations for the random measure associated with the jumps of $\mathbf{X}$. In general terms, suppose that the Poisson random measure N associated with the jumps of $\mathbf{X}$, namely

$$N(B) = \#\left\{t > 0 : (t, \mathbf{X}(t) - \mathbf{X}(t^-)) \in B\right\}, \tag{5}$$

has the representation

$$N\left(\cdot\right) = \sum_{i=1}^{\infty} \delta_{(U_i,\mathbf{J}_i)}\left(\cdot\right), \tag{6}$$

when restricted to $[0,T] \times \mathbb{R}_0^d$, where here $\{U_i\}_{i\geq 1}$ is a sequence of i.i.d uniform random variables on $[0,T]$ and $\{\mathbf{J}_i\}_{i\geq 1}$ is a sequence of random vectors. Then, $\mathbf{X}$ has the *shot-noise series* representation

$$\mathbf{X}(t) = \mathbf{b}t + \sum_{i=1}^{\infty} \left[\mathbf{J}_i\mathbf{1}(U_i \leq t) - t\mathbf{c}_i\right], \quad 0 \leq t \leq T, \tag{7}$$

for suitable centers $\mathbf{c}_i$ that compensate the jumps. The random variables U's govern the times of the jumps, while the $\mathbf{J}$'s give the size of the jumps.

The general idea behind the representation (6) is as follows. Think of the jumps $\mathbf{J}$'s as random responses to fictitious "shots" occurring in the past according to a homogeneous Poisson process. Let $0 < \Gamma_1 < \Gamma_2 < \ldots$ be the arrival times of a homogeneous Poisson process on $(0, \infty)$ with constant intensity T. Denote $\mathbf{J}_i$ the jump originating as consequence of a shot occurring Γ_i time units ago. The distribution of the jumps are dictated by a probability measure $\sigma(u; \cdot)$ on $\mathcal{B}(\mathbb{R}^d)$ in such a way that

$$\mathbb{P}\left(\mathbf{J}_i \in B \mid \{\Gamma_j\}_{j \geq 1}, \{\mathbf{J}_j\}_{j \neq i}\right) = \sigma\left(\Gamma_i, B\right), \quad B \in \mathcal{B}(\mathbb{R}_0^d). \tag{8}$$

It follows that the jumps $\mathbf{J}_1, \ \mathbf{J}_2, \ldots$ form a Poisson process in $\mathbb{R}^d$ with mean measure $T \int_0^\infty \sigma(u; B) du$, (*Why?*). Consequently, the point process $\sum_{i=1}^\infty \delta_{(U_i, \mathbf{J}_i)}(\cdot)$, where $\{U_i\}_i$ are i.i.d. uniform r.v. on $[0, T]$ and independent from $\{\mathbf{J}_i\}_i$, is a Poisson point process with mean measure of the form

$$m(dt, d\mathbf{x}) = \int_0^\infty \sigma(u; d\mathbf{x})\, du\, dt.$$

Therefore, if the Lévy measure of the Lévy process has the representation:

$$\nu(B) = \int_0^\infty \sigma(u; B) du, \tag{9}$$

then the measure N in (5), when restricted to $[0, T] \times \mathbb{R}_0^d$, has the same law as $\sum_{i=1}^\infty \delta_{(U_i, \mathbf{J}_i)}(\cdot)$. There are other considerations to think about for the representation (7) to hold true. It has to do with the probabilistic structure of the jumps $\{\mathbf{J}_i\}$. Roughly speaking, to avoid divergence problems and to guarantee the existence of compensating centers $\mathbf{c}_i$, it is necessary that the magnitude of the jumps decreases as the elapsed time between the jump and the shot increases (an appealing physical assumption as well). This is better explained if we notice that (8) is equivalent to having

$$\mathbf{J}_i = \mathbf{H}\left(\Gamma_i, \mathbf{V}_i\right),$$

for a sequence of random elements $\{\mathbf{V}_i\}_{i \geq 1}$ independent of $\{U_i, \Gamma_i\}_{i \geq 1}$ (see Lemma 2.22 of [1]). Then, $\|\mathbf{H}(r, \mathbf{v})\|$ should decrease in r for (7) to be true. Notice that

$$\mathbf{H}\left(u, \mathbf{V}_i\right) \sim \sigma(u; d\mathbf{x}).$$

Let us summarize the conditions and give the main theorem for the simulation of Lévy processes.

Condition 1 *The jump measure of* $\mathbf{X}$ *can be written as*

$$N(\cdot) \overset{\mathfrak{D}}{=} \sum_{i=1}^{\infty} \delta_{(U_i, \mathbf{H}(\Gamma_i, \mathbf{V}_i))}(\cdot). \tag{10}$$

for a function $\mathbf{H} : (0, \infty) \times S \to \mathbb{R}^d$, *where* S *is an arbitrary measurable space. Here,* $\{\Gamma_i\}_{i=1}^\infty$ *is a homogeneous Poisson process on* $\mathbb{R}_+$ *with intensity* T, $\{U_i\}_{i=1}^\infty$ *is an independent random sample with uniform distribution on* $(0, T)$, *and* $\{\mathbf{V}_i\}_{i=1}^\infty$ *is an independent random sample* $\{\mathbf{V}_i\}_{i=1}^\infty$ *with common distribution* F *on the space* S.

Condition 2 *For any Poisson process* $\{\Gamma_i^1\}_{i=1}^{\infty}$ *on* $\mathbb{R}_+$ *with unit rate,*

$$\mathbf{A}(\Gamma_n^1) - \mathbf{A}(n) \to 0 \;\; a.s., \tag{11}$$

where

$$\mathbf{A}(s) \equiv \int_0^s \int_S \mathbf{H}(r, \mathbf{v}) \mathbf{1} \left(\|\mathbf{H}(r, v)\| \leq 1 \right) F(d\mathbf{v}) dr. \tag{12}$$

The next lemma gives sufficient conditions for (11) (see pp. 409 [3]):

Lemma 1 *The limit in (11) holds true if either one of the following conditions is satisfied:*
i) $\mathbf{a} \equiv \lim_{s\to\infty} \mathbf{A}(s)$ *exists in* $\mathbb{R}^d$*;*
ii) *the mapping* $r \to \|\mathbf{H}(r, v)\|$ *is nonincreasing for each* $v \in S$*.*

The following result establishes the series representations for Lévy processes.

Proposition 1 *If the conditions 1 and 2 are satisfied then, a.s.*

$$\mathbf{X}(t) \overset{\mathfrak{D}}{=} \mathbf{b}t + \sum_{i=1}^{\infty} \left[\mathbf{H}(\Gamma_i, \mathbf{V}_i) \mathbf{1} \left(U_i \leq t \right) - t\mathbf{c}_i \right], \tag{13}$$

for all $t \in [0, T]$*, where* $\mathbf{c}_i \equiv \mathbf{A}(i) - \mathbf{A}(i-1)$*.*

Proof: Notice that the Lévy-Itô representation (1) takes the form

$$\mathbf{X}(t) = \mathbf{b}t + \int_{[0,t]\times\mathbb{R}_0^d} \mathbf{x}\mathbf{1}(\|\mathbf{x}\| \leq 1)(\mathcal{J} - E\mathcal{J})(du, d\mathbf{x}) + \int_{[0,t]\times\mathbb{R}_0^d} \mathbf{x}\mathbf{1}(\|\mathbf{x}\| > 1)\mathcal{J}(du, d\mathbf{x}).$$

Define

$$M(\cdot) \equiv \sum_{i=1}^{\infty} \delta_{(U_i, \Gamma_i, \mathbf{V}_i)}(\cdot).$$

From Proposition 3.8. of [2], M is a (marked) Poisson process on $R \equiv [0, T] \times \mathbb{R}_+ \times \mathbb{R}_0^d$ with mean measure $du\, dr\, F(d\mathbf{v})$. By a "change of variables",

$$\begin{aligned} \mathbf{X}(t) = \mathbf{b}t &+ \int_{R_t} \mathbf{H}(r, \mathbf{v})\mathbf{1}(\|\mathbf{H}(r, \mathbf{v})\| \leq 1)(M(du, dr, d\mathbf{v}) - dudrF(d\mathbf{v})) \\ &+ \int_{R_t} \mathbf{H}(r, \mathbf{v})\mathbf{1}(\|\mathbf{H}(r, \mathbf{v})\| > 1)(M(du, dr, d\mathbf{v}) - dudrF(d\mathbf{v})), \end{aligned}$$

where $R_t \equiv [0, t] \times \mathbb{R}_+ \times \mathbb{R}_0^d$. Define

$$\begin{aligned} \mathbf{X}_s(t) = \mathbf{b}t &+ \int_{R_t^s} \mathbf{H}(r, \mathbf{v})\mathbf{1}(\|\mathbf{H}(r, \mathbf{v})\| \leq 1)(M(du, dr, d\mathbf{v}) - dudrF(d\mathbf{v})) \\ &+ \int_{R_t^s} \mathbf{H}(r, \mathbf{v})\mathbf{1}(\|\mathbf{H}(r, \mathbf{v})\| > 1)(M(du, dr, d\mathbf{v}) - dudrF(d\mathbf{v})), \end{aligned}$$

where $R_t^s \equiv [0, t] \times [0, s] \times \mathbb{R}_0^d$. Using that the Poisson process M is an independently scatter measure (that is, $M(A_1), \ldots, M(A_n)$ are mutually independent for disjoint sets $A_1, \ldots, A_n$), we can verify in a standard way that $\mathbf{X}_s(t)$ has independent increments both with respect to $s \in [0, \infty)$ and $t \in [0, T]$. Also, notice that

$$\mathbf{X}_s(t) = \mathbf{b}t + \sum_{i:\Gamma_i \leq s} \mathbf{H}(\Gamma_i, \mathbf{V}_i)\mathbf{1}\left(U_i \leq t\right) - t\mathbf{A}(s), \tag{14}$$

implying that $\mathbf{X}_s(t)$ enjoys càdlàg paths in s for each t. We claim that almost surely,

$$\lim_{s \to \infty} \mathbf{X}_s(t) = \mathbf{X}(t),$$

for all $t \in [0, T]$. Fix $t \in [0, T]$ and take a sequence $s_n \uparrow \infty$. Since $\mathbf{X}_\cdot(t)$ has càdlàg paths, it suffices to check that $\lim_{n\to\infty} \mathbf{X}_{s_n}(t) = \mathbf{X}(t)$ a.s. Furthermore, since $\mathbf{X}_{s_n}(t) = \sum_{i=1}^{n}(\mathbf{X}_{s_i}(t) - \mathbf{X}_{s_{i-1}}(t))$ and since $\mathbf{X}_\cdot(t)$ has independent increment, it is enough to have convergence is in distribution. The later can be deduced from arguments based on characteristic function. □

Remark 1 *We finally point out that if condition (11) is true, and the representation (10) holds in distribution, then the representation (13) is valid in the sense of finite dimensional distributions. Representation (10) can be obtained in law if the Lévy measure has the decomposition*

$$\nu(B) = \int_0^\infty \sigma(u; B) du, \tag{15}$$

where $\sigma(u; B) = \mathbb{P}\left[\mathbf{H}(u, \mathbf{V}) \in B\right]$.

Remark 2 *The series (13) simplifies further when* $\int_{\|\mathbf{x}\| \leq 1} \|\mathbf{x}\|\nu(d\mathbf{x}) < \infty$, *namely, when* $\mathbf{X}$ *has paths of bounded variation a.s. (see Theorem 21.9 of [4]). Concretely, a.s.*

$$\mathbf{X}(t) = (\mathbf{b} - \mathbf{a})\, t + \sum_{i=1}^{\infty} \mathbf{J}_i \mathrm{I}\left(U_i \leq t\right), \tag{16}$$

where

$$\mathbf{a} = \int_{\|\mathbf{x}\| \leq 1} \mathbf{x}\nu(d\mathbf{x}).$$

The vector $\mathbf{d} \equiv \mathbf{b} - \mathbf{a}$ *is called the* drift *of the Lévy process.*

5 “Density Transformation” of Lévy processes

Theorem 2 *Let* X *be a Lévy process with Lévy triple* (b, c^2, ν) *under some probability measure* $\mathbb{P}$. *Then the following two conditions are equivalent:*

- *(a) There is a probability measure* $\mathbb{Q} \overset{\text{loc}}{\sim} \mathbb{P}$ *such that* X *is a Lévy process with triplet* (b', c'^2, ν') *under* $\mathbb{Q}$.
- *(b) All the following conditions hold.*
  - *(i)* $\nu'(dx) = k(x)\nu(dx)$, *for some Borel function* $k : \mathbb{R} \to (0, \infty)$.

(ii) $b' = b + \int x 1_{|x|<1}(k(x) - 1)\nu(dx) + c\beta$, *for some* $\beta \in \mathbb{R}$.

(iii) $c' = c$.

(iv) $\int \left(1 - \sqrt{k(x)}\right) \nu(dx) < \infty$.

Theorem 3 *Suppose that the equivalent conditions of the previous theorem are satisfied. Then, the density process* $\{\xi_t : 0 \leq t \leq T\}$, *defined by* $\xi_t \equiv \frac{d\mathbb{Q}_t}{d\mathbb{P}_t}$, *is given by the formula*

$$\xi_t \equiv \exp\Bigg(\beta W_t - \frac{1}{2}\beta^2 t + \lim_{\varepsilon \downarrow 0}\left(\int_0^t \int_{|x|>\varepsilon} \log k(x) N(ds, dx) - t \int_{|x|>\varepsilon} (k(x) - 1)\nu(dx)\right)\Bigg),$$

with $\mathbb{E}_{\mathbb{P}}[\xi_t] \equiv 1$. *The convergence on the right-hand side of the formula above is uniform in* t *on any bounded interval.*

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