SMALL-TIME EXPANSIONS OF THE DISTRIBUTIONS, DENSITIES, AND OPTION PRICES OF STOCHASTIC VOLATILITY MODELS WITH LÉVY JUMPS

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ABSTRACT. We consider a stochastic volatility model with Lévy jumps for a log-return process $Z = (Z_t)_{t\geq 0}$ of the form $Z = U + X$, where $U = (U_t)_{t\geq 0}$ is a classical stochastic volatility process and $X = (X_t)_{t\geq 0}$ is an independent Lévy process with absolutely continuous Lévy measure ν. Small-time expansions, of arbitrary polynomial order in time t, are obtained for the tails $\mathbb{P}(Z_t \geq z)$, $z > 0$, and for the call-option prices $\mathbb{E}\left(e^{z+Z_t} - 1\right)_+$, $z \neq 0$, assuming smoothness conditions on the Lévy density away from the origin and a small-time large deviation principle on U. Our approach allows for a unified treatment of general payoff functions of the form $\varphi(x)\mathbf{1}_{x\geq z}$ for smooth functions φ and $z > 0$. As a consequence of our tail expansions, the polynomial expansions in t of the transition densities f_t are also obtained under mild conditions.

1. Introduction

It is generally recognized that accurate modeling of the option market and asset prices requires a mixture of a continuous diffusive component and a jump component. For instance, based on high-frequency statistical methods for Itô semimartingales, several empirical studies have rejected statistically the null hypothesis of either a purely-jump or a purely-continuous model (see, e.g., [3], [4], [5], [31]). Similarly, by characterizing the small-time behavior of at-the-money (ATM) and out-of-the-money (OTM) call option prices, [9] argued that both, continuous and jump, components are necessary to explain the implied volatilities behavior of S&P500 index options. Historically, local volatility models (and more recently stochastic volatility models) were the models of choice to replicate the skewness of the market implied volatilities at a given time (see [19] and [21] for more details). However, it is a well-known empirical fact that implied volatility skewness is more dramatic as the expiration time approaches. Such a phenomenon is hard to reproduce within the purely-continuous framework unless the "volatility of volatility" is forced to take very high values. Furthermore, as it is nicely explained in [11] (Chapter 1), the very existence of a market for short-term options is evidence that option market participants operate under the assumption that a jump component is present.

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In recent years the literature of small-time asymptotics for vanilla option prices of jump-diffusion models has grown significantly with strong emphasis to consider either a purely-continuous model or a purely Lévy model. In the case of stochastic volatility models and local volatility models, we can mention, among others, [6], [7], [12], [17], [18], [20], [22], [32]. In the case of Lévy process, [35] and [38] show independently that OTM option prices are generally[1] asymptotically equivalent to the time-to-maturity τ as $\tau \to 0$. In turn, such a behavior implies that the implied volatilities of a Lévy model explodes as $\tau \to 0$. The exact first order asymptotic behavior of the implied volatility close to maturity was independently obtained by [14] and [38], while the former paper also gives the second order asymptotic behavior. There are few pieces of work that consider simultaneously stochastic volatility and jumps in the model. One such work is [9] which obtains, partially by heuristic arguments, the first order asymptotic behavior of an Itô semimartingale with jumps. Concretely, ATM option prices of pure-jump models of bounded variation decrease at the order $O(\tau)$, while they are just $O(\sqrt{\tau})$ under the presence of a Brownian component. By considering a stable pure-jump component, they also show that, in general, the behavior could be $O(\tau^{\beta})$ for some $\beta \in (0,1)$. For OTM options, they also argue that the first order behavior is $O(e^{-c/\tau})$ in the purely-continuous case, while it behaves like $O(\tau)$ under the presence of jumps. Very recently, [30] formally shows that the leading term of ATM option prices is of order $\sqrt{T}$ for a relatively general class of purely-continuous Itô models, while for a more general type of Itô processes with α-stable-like small jumps, the leading term is $O(\tau^{1/\alpha})$ (see also [14, Proposition 4.2] and [38, Proposition 5] for related results in exponential Lévy models). Fractional expansions are also obtained for the distribution functions of some Lévy processes in [28].

In this article, we consider a jump diffusion model by combining a stochastic volatility model with a pure-jump Lévy process. More precisely, let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, on which we consider a risk-free asset with constant interest rate $r \geq 0$ and a risky asset with price process

$$S_t := S_0 e^{rt+Z_t}. \tag{1.1}$$

For the log-return process $Z = (Z_t)_{t\geq 0}$, we consider the following jump diffusion model:

$$Z_t = U_t + X_t, \quad dU_t = \mu(Y_t)dt + \sigma(Y_t)dW_t^{(1)}, \tag{1.2}$$

$$dY_t = \alpha(Y_t)dt + \gamma(Y_t)dW_t^{(2)} \tag{1.3}$$

with $U_0 = X_0 = 0$, $Y_0 = y_0 \in \mathbb{R}$. Here, $W^{(1)}$ and $W^{(2)}$ are Wiener processes adapted to $(\mathcal{F}_t)$, X is an independent $(\mathcal{F}_t)$-adapted pure-jump Lévy process with triplet $(b, 0, \nu)$, and σ, γ, μ and α are deterministic functions such that (1.2)-(1.3) admits a solution. In order for $(e^{-rt}S_t)_{t\geq 0}$ to be a $\mathbb{Q}$-martingale, we also assume that $\int_{z>1} e^z \nu(dz) < \infty$ and also

$$b = -\int_{\mathbb{R}} \left(e^z - 1 - z\mathbf{1}_{|z|\leq 1}\right) \nu(dz), \quad \text{and} \quad \mu(y) = -\frac{1}{2}\sigma^2(y). \tag{1.4}$$

Note that the model (1.2)-(1.3) incorporates not only jumps in the asset price process but also volatility clustering and leverage effects.

[1]That is, except for some pathological cases (see [35] for examples)

For $z \neq 0$ and $t > 0$, let

$$G_t(z) := \mathbb{E}\left(e^{z+Z_t} - 1\right)_+, \tag{1.5}$$

where $\mathbb{E}$ denotes, from now on, the expectation under a fixed risk-neutral probability measure $\mathbb{Q}$. We will show that, under mild conditions, the following small-time expansions for $G_t(z)$ hold true:

$$G_t(z) = \sum_{j=0}^{n} b_j(z)\,\frac{t^j}{j!} + O(t^{n+1}), \tag{1.6}$$

for each $n \geq 0$ and certain functions b_j. Note that the time-t price of a European call option with strike K, which is not at-the-money, can then be expressed as

$$C(t,s) := e^{-r(T-t)}\mathbb{E}\left((S_T - K)_+ \middle| S_t = s\right) = Ke^{-r\tau}G_\tau(\ln s - \ln K), \tag{1.7}$$

where $\tau = T - t$ and $s \neq K$. Hence, the small-time behavior of (1.5) leads to close-to-expiry approximations for the price of an arbitrary not-at-the-money call option as a polynomial expansion in time-to-maturity.

Our method of proof is built on a type of iterated Dynkin formula of the form

$$\mathbb{E}g(X_t) = g(0) + \sum_{k=1}^{n} \frac{t^k}{k!} L^k g(0) + \frac{t^{n+1}}{n!}\int_0^1 (1-\alpha)^n \mathbb{E}\left(L^{n+1}g(X_{\alpha t})\right) d\alpha, \tag{1.8}$$

where g is a sufficiently smooth function and L is the infinitesimal generator of the Lévy process X. The main complication with option call prices arises from the lack of smoothness of the payoff function $g_z(x) = (e^{z+x} - 1)_+$. In order to "regularize" the payoff function g, we follow a two step procedure. First, we decompose the Lévy process into a compound Poisson process with a smooth jump density vanishing in a neighborhood of the origin and an independent Lévy process with small jumps. Then, we condition the expectation $\mathbb{E}g(X_t)$ on the number of jumps of the compound Poisson component of X and apply Dynkin's formula on each of the resulting terms. Contrary to the approaches in [14] and [38], where the special form of the payoff function $g_z(x) = (e^{z+x} - 1)_+$ plays a key role, our approach can handle more general payoff functions of the form

$$g_z(x) = \varphi(x)\mathbf{1}_{\{x \geq z\}}, \tag{1.9}$$

for a smooth function φ and positive z. By considering the particular case $\varphi(x) \equiv 1$, we generalize the distribution expansions in [13] to our jump-diffusion setting. Let us emphasize that the process Z in (1.2) is not truly a Markov model but rather a hidden Markov model. This fact causes some technical subtleties that require a careful analysis of the iterated infinitesimal generator of the bivariate Markov process $\{(U_t, Y_t)\}_{t \geq 0}$.

As an equally relevant second contribution of our paper, we also obtain polynomial expansions for the transition densities f_t of the Lévy process, under conditions involving only the Lévy density of X. This is an important improvement to our former results in [13], where a uniform boundedness condition on all the derivatives of f_t away from the origin was imposed. Expansions for the transition densities of local volatility models (with possibly finite-jump activity) have appeared before in the literature (e.g. see [1], [2], [39]). Unlike our approach, the general idea in the referred papers consists of first proposing the general form of the expansion, and then choosing the coefficients so that either the backward or forward Kolmogorov equation is satisfied. The resulting coefficients typically involve iterated infinitesimal generators as in ours expansions, even though our approximations are uniform away from the origin.

The paper is organized as follows. Section 2 contains some preliminary results on Lévy processes, which will be needed throughout the paper. Section 3 establishes the small-time expansions, of arbitrary polynomial order in t, for both the tail distributions $\mathbb{P}(Z_t \geq z)$, $z > 0$, and the call-option price function $G_t(z)$, $z \neq 0$. This section also justifies the validity of our results for payoff functions of the form (1.9). Section 4 illustrates the first few terms of those expansions. Interestingly enough, the first two coefficients of the expansion of the general model coincide with the first two coefficients of an exponential Lévy model. Section 5 obtains the asymptotic behavior of the corresponding implied volatility. Section 6 gives a small-time expansion for the transition density of a general Lévy process under rather mild conditions. The proofs of our main results are deferred to Appendices.

2. Background and preliminary results

2.1. **Notation.**

Throughout this paper, C^n or $C^n(\mathbb{R})$, $n \geq 0$, is the class of real valued functions, defined on $\mathbb{R}$, which have continuous derivatives of order $0 \leq k \leq n$. $C_b^n \subset C^n$ corresponds to the ones having bounded derivatives. In a similar fashion, C^∞ or $C^\infty(\mathbb{R})$ is the class of real valued function, defined on $\mathbb{R}$, which have continuous derivatives of any order $k \geq 0$, while $C_b^\infty(\mathbb{R}) \subset C^\infty$ are again the ones having bounded derivatives. Sometimes $\mathbb{R}$ will be replaced by $\mathbb{R}\setminus\{0\}$ or $\mathbb{R}^k$ when the functions are defined on these spaces.

Throughout this section, let X be a general Lévy process with triplet (b, σ^2, ν) defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$. Let us write X in terms of its Lévy-Itô decomposition:

$$X_t = bt + \sigma W_t + \int_0^t \int_{|z|>1} z\mu(ds, dz) + \int_0^t \int_{|z|\leq 1} z\bar{\mu}(ds, dz),$$

where W is a Wiener process and μ is an independent Poisson measure on $\mathbb{R}_+ \times \mathbb{R}\backslash\{0\}$ with mean measure $dt\nu(dz)$ and compensator $\bar{\mu}$. For each $\varepsilon > 0$, let $c_\varepsilon \in C^\infty$ be a symmetric truncation function such that $\mathbf{1}_{[-\varepsilon/2,\varepsilon/2]}(z) \leq c_\varepsilon(z) \leq \mathbf{1}_{[-\varepsilon,\varepsilon]}(z)$ and consider the processes:

$$\bar{X}_t^\varepsilon := \int_0^t \int_{\mathbb{R}} z\, \bar{c}_\varepsilon(z)\mu(ds, dz), \qquad \text{and} \qquad X_t^\varepsilon := X_t - \bar{X}_t^\varepsilon, \tag{2.1}$$

where $\bar{c}_\varepsilon(x) := 1 - c_\varepsilon$. Clearly, $\bar{X}^\varepsilon$ is a compound Poisson process with intensity of jumps $\lambda_\varepsilon := \int \bar{c}_\varepsilon(z)\nu(dz)$, and jumps distribution $\bar{c}_\varepsilon(z)\nu(dz)/\lambda_\varepsilon$. Throughout, ξ_i^ε stands for the jumps of the process $\bar{X}^\varepsilon$. Note that the remaining process X^ε has infinitesimal generator L_ε given by

$$L_\varepsilon g(y) = b_\varepsilon g'(y) + \frac{\sigma^2}{2} g''(y) + \int \left\{ g(y+z) - g(y) - zg'(y)\mathbf{1}_{|z|\leq 1} \right\} c_\varepsilon(z)\nu(dz), \tag{2.2}$$

for $g \in C_b^2$, where

$$b_\varepsilon := b - \int_{|z|\leq 1} z\bar{c}_\varepsilon(z)\nu(dz).$$

The following tail estimate for X^ε is also used in the sequel:

$$\mathbb{P}(|X_t^\varepsilon| \geq z) \leq t^{az} \exp(az_0 \ln z_0) \exp(az - az \ln z), \tag{2.3}$$

where $a \in (0, \varepsilon^{-1})$, and $t, z > 0$ satisfy $t < z/z_0$ for some z_0 depending only on a (see [37, Section 2.6], [36, Lemma 3.2] and [13, Remark 3.1] for proofs and extensions).

Throughout the paper, we also make the following standing assumptions:

$$\nu(dz) = s(z)dz,\ s \in C^\infty(\mathbb{R}\backslash\{0\}) \text{ and } \gamma_{k,\delta} := \sup_{|z|>\delta} |s^{(k)}(z)| < \infty, \forall \delta > 0, \tag{2.4}$$

$$\int_{|z|>1} e^{c|z|}\nu(dz) < \infty, \text{ for some } c > 2. \tag{2.5}$$

Finally, the following terminology will also be needed:

$$s_\varepsilon := c_\varepsilon s, \quad \bar{s}_\varepsilon := (1 - c_\varepsilon)s, \quad L^0 g = g, \quad L^{k+1}g = L(L^k g), \quad (k \geq 0),$$

$$\bar{s}_\varepsilon^{*0} * g = g, \quad \bar{s}_\varepsilon^{*1} = \bar{s}_\varepsilon, \quad \bar{s}_\varepsilon^{*k}(x) = \int \bar{s}_\varepsilon^{*(k-1)}(x-u)\bar{s}_\varepsilon(u)du, \quad (k \geq 2).$$

2.2. Dynkin's formula for smooth subexponential functions.

Let us recall that taking expectations in the well-known Dynkin's formula gives:

$$\mathbb{E}g(X_t) = g(0) + \int_0^t \mathbb{E}\left(Lg(X_u)\right)du = g(0) + t\int_0^1 \mathbb{E}\left(Lg(X_{\alpha t})\right)d\alpha, \tag{2.6}$$

valid if $g \in C_b^2$. Iterating (2.6), one obtains the following expansion for $g \in C_b^{2n+2}$ (e.g., see [23, Proposition 9]):

$$\mathbb{E}g(X_t) = g(0) + \sum_{k=1}^{n} \frac{t^k}{k!} L^k g(0) + \frac{t^{n+1}}{n!}\int_0^1 (1-\alpha)^n \mathbb{E}\left(L^{n+1}g(X_{\alpha t})\right)d\alpha. \tag{2.7}$$

For our purposes, it will be useful to extend (2.7) to subexponential functions. We have the following result whose proof can be found in Appendix A:

Proposition 2.1. *Let ν satisfy (2.5), and let $g \in C^{2n+2}$ be such that*

$$\limsup_{|y|\to\infty} e^{-\frac{c}{2}|y|}|g^{(i)}(y)| < \infty, \tag{2.8}$$

for any $0 \leq i \leq 2n+2$. Then, (2.7) holds true.

In order to work with the iterated infinitesimal generator L^k appearing in (2.7), the forthcoming representation will turn out to be useful (see [13, Lemma 4.1] for its verification[2]). Set

$$b_0 := -\int_{\mathbb{R}} \bar{c}_\varepsilon(u)\nu(du), \quad b_1 := b - \int_{\mathbb{R}} u(c_\varepsilon(u) - \mathbf{1}_{|u|\leq 1})\nu(du),$$

$$b_2 := \sigma^2/2, \quad b_3 := \frac{1}{2}\int_{\mathbb{R}} u^2 c_\varepsilon(u)\nu(du), \quad \text{and} \quad b_4 := \int_{\mathbb{R}} \bar{c}_\varepsilon(u)\nu(du),$$

and note that all these constants depend on $\varepsilon > 0$, but this is not explicitly indicated for the ease of notation.

[2]Note that for convenience we switch the role of b_3 and b_4.

Lemma 2.2. *Let* $\mathcal{K}_k = \{\mathbf{k} = (k_0, \dots, k_4) \in \mathbb{N}^5 : k_0 + \dots + k_4 = k\}$ *and for* $\mathbf{k} \in \mathcal{K}_k$*, let* $\ell_{\mathbf{k}} := k_1 + 2k_2 + 2k_3$*. Then, for any* $k \geq 1$ *and* $\varepsilon > 0$*,*

$$L^k g(x) = \sum_{\mathbf{k}\in\mathcal{K}_k} \prod_{i=0}^{4} b_i^{k_i} \binom{k}{\mathbf{k}} B_{\mathbf{k},\varepsilon} g(x), \tag{2.9}$$

where

$$B_{\mathbf{k},\varepsilon} g(x) := \begin{cases} \int g^{(\ell_{\mathbf{k}})} \left(x + \sum_{j=1}^{k_3} \beta_j w_j + \sum_{i=1}^{k_4} u_i \right) d\pi_{\mathbf{k},\varepsilon}, & \text{if } k_3 + k_4 > 0, \\ g^{(\ell_{\mathbf{k}})}(x), & \text{if } k_3 = k_4 = 0, \end{cases}$$

and the above integral is with respect to the probability measure

$$d\pi_{\mathbf{k},\varepsilon} = \prod_{j=1}^{k_3} \frac{1}{b_3} c_\varepsilon(w_j) w_j^2 \nu(dw_j)(1-\beta_j) d\beta_j \prod_{i=1}^{k_4} \frac{1}{b_4} \bar{c}_\varepsilon(u_i)\nu(du_i),$$

on $\mathbb{R}^{k_3} \times [0,1]^{k_3} \times \mathbb{R}^{k_4}$ *(under the standard conventions that* $0/0 = 1$ *and* $\prod_{i=1}^{0} = 1$*).*

Remark 2.3. The expansion (2.9) holds true for (possibly unbounded) functions $g \in C^{2k+2}$ satisfying (2.8) for any $0 \leq i \leq 2k+2$.

3. Small-time expansions for the tail distributions and option prices

In this section, we derive the small-time expansions for both the tail distribution $\mathbb{P}(Z_t \geq z)$, $z > 0$, and for the call-option price function $\mathbb{E}\left(e^{z+Z_t} - 1\right)_+$, $z \neq 0$. With an approach similar to that in [13, Theorem 3.2], the idea is to apply the following general moment expansion (easily obtained by conditioning on the number of jumps of the process $\bar{X}_t^\varepsilon$ introduced in (2.1)):

$$\mathbb{E}f(Z_t) = e^{-\lambda_\varepsilon t}\mathbb{E}f\left(U_t + X_t^\varepsilon\right) + e^{-\lambda_\varepsilon t} \sum_{k=n+1}^{\infty} \frac{(\lambda_\varepsilon t)^k}{k!} \mathbb{E}f\left(U_t + X_t^\varepsilon + \sum_{i=1}^{k} \xi_i^\varepsilon\right) \tag{3.1}$$

$$+ e^{-\lambda_\varepsilon t} \sum_{k=1}^{n} \frac{(\lambda_\varepsilon t)^k}{k!} \mathbb{E}f\left(U_t + X_t^\varepsilon + \sum_{i=1}^{k} \xi_i^\varepsilon\right), \tag{3.2}$$

where ξ_i^ε are the jumps of the process $\bar{X}^\varepsilon$. We shall take $f(u) = f_z(u) := \mathbf{1}_{\{u \geq z\}}$ in order to obtain the expansion of the transition distribution and $f(u) = f_z(u) := (e^{z+u} - 1)_+$ in order to obtain the expansion of the call-option price. To work out the terms in (3.2), we use the iterated formula (2.7), while to estimate the terms in (3.1), we assume that the underlying stochastic volatility model U satisfies a small-time large deviation principle:

$$\lim_{t\to 0} t \ln \mathbb{P}(U_t > u) = -\frac{1}{2} d(u)^2, \quad (u > 0), \tag{3.3}$$

where $d(u)$ is a strictly positive rate function. In Section 3.4 we review conditions for (3.3) to hold.

3.1. **Expansions for the tail distributions.**

We first treat the case $f_z(u) := \mathbf{1}_{\{u\geq z\}}$. We have the following expansion for the tail distributions of Z (its proof can be found in Appendix B):

Theorem 3.1. *Let $z_0 > 0$, $n \geq 1$, and $0 < \varepsilon < z_0/(n+1) \wedge 1$. Let the dynamics of Z be given by (1.2) and (1.3), and the conditions (2.4)-(2.5) and (3.3) be satisfied. Then, there exists $t_0 > 0$ such that, for any $z \geq z_0$ and $0 < t < t_0$,*

$$\mathbb{P}(Z_t \geq z) = e^{-\lambda_\varepsilon t} \sum_{j=1}^{n} \widehat{A}_{j,t}(z) \frac{t^j}{j!} + O_{\varepsilon,z_0}(t^{n+1}), \tag{3.4}$$

where

$$\widehat{A}_{j,t}(z) := \sum_{k=1}^{j} \binom{j}{k} \mathbb{E}\left\{ \left(L_\varepsilon^{j-k} \widehat{f}_{k,z} \right)(U_t) \right\}$$

with $\widehat{f}_{k,z}(y) := \int_{z-y}^{\infty} \bar{s}_\varepsilon^{*k}(u)du$.

The expression (3.4) is not really satisfactory since the coefficients $\widehat{A}_{j,t}$ are time-dependent and so the asymptotic behaviors as $t \to 0$ are unclear. In order to obtain an expansion of $\widehat{A}_{j,t}$, we can further apply an iterated expansion for $\mathbb{E}g(U_t, Y_t)$. Indeed, assuming for simplicity that $W^{(1)}$ and $W^{(2)}$ are independent, (U, Y) is a Markov process with infinitesimal generator

$$\mathcal{L}g(u, y) = \mu(y)\frac{\partial g}{\partial u} + \frac{\sigma^2(y)}{2}\frac{\partial^2 g}{\partial u^2} + \alpha(y)\frac{\partial g}{\partial y} + \frac{\gamma^2(y)}{2}\frac{\partial^2 g}{\partial y^2}, \tag{3.5}$$

for $g \in C_b^2$. Itô's formula and induction imply that

$$\mathbb{E}g(U_t, Y_t) = g(u_0, y_0) + \sum_{k=1}^{n} \frac{t^k}{k!}\mathcal{L}^k g(u_0, y_0) + \frac{t^{n+1}}{n!}\int_0^1 (1-\alpha)^n \mathbb{E}\left\{\mathcal{L}^{n+1} g(U_{\alpha t}, Y_{\alpha t})\right\} d\alpha, \tag{3.6}$$

for any function g such that $\mathcal{L}^k g(u, y)$ is well-defined and belongs to C_b for $0 \leq k \leq 2n+2$. As in the case of the infinitesimal generator of X, one can view the operator (3.5) as the sum of four operators. However, given that in general those operators do not commute, it is not possible to write a simple closed-form expression for $\mathcal{L}^k g(u_0, y_0)$ as in the case for X. Nevertheless, the following result gives a recursive method to get such expression when $\mu(y) = -\sigma^2(y)/2$ and $g(u, y) = h(u)$ as is needed here:

Proposition 3.2. *Let the dynamics of U and Y be given by (1.2) and (1.3) with independent $W^{(1)}$ and $W^{(2)}$ and with C^∞ deterministic functions α, σ^2, and γ^2. On $C^2(\mathbb{R}^2)$, let:*

$$\mathcal{L}_u h(u) := h''(u) - h'(u), \quad \mathcal{L}_y \tilde{h}(y) := \frac{\gamma^2(y)}{2}\tilde{h}''(y) + \alpha(y)\tilde{h}'(y).$$

Let $h \in C^{2n+2}$, and fix $g(u, y) := h(u)$. Then, the infinitesimal generator (3.5) is such that

$$\mathcal{L}^k g(u, y) = \sum_{j=0}^{k} B_j^k(y)\mathcal{L}_u^j h(u), \quad \textit{for} \quad k \geq 0,$$

where $B_j^k(y)$ are defined iteratively as follows:

$$B_0^0(y)=1,\quad B_j^k(y)=0,\quad \forall j\notin\{0,\dots,k\},$$
$$B_j^k(y)=\mathcal{L}_y B_j^{k-1}(y)+\frac{\sigma^2(y)}{2}B_{j-1}^{k-1}(y),\quad 0\le j\le k,\quad k\ge 1.$$

Proof. The proof is done by induction. □

Using the previous result, we can easily check conditions for the iterated formula (3.6) to hold. To this end, let us define the following class of functions:

$$\mathcal{C}_l^n=\left\{p\in C^n:|p^{(i)}(x)|\le\mathcal{M}_n(1+|x|),\ \text{for some } \mathcal{M}_n<\infty \text{ independent of } x,\ 0\le i\le n\right\}.$$

Corollary 3.3. *In addition to the conditions of Proposition 3.2, let $\gamma\in\mathcal{C}_l^0$ and let $\alpha,\sigma^2,\gamma^2\in\mathcal{C}_l^k$, for any $k\ge 0$. Then, (3.6) is satisfied for $g(u,y):=h(u)$ whenever $h\in C_b^{2n+2}$.*

Proof. Using Itô's formula and induction, we can show (3.6) provided that

$$\int_0^t\frac{\partial\mathcal{L}^n g(U_s,Y_s)}{\partial u}\sigma(Y_s)dW_s^{(1)},\quad\text{and}\quad\int_0^t\frac{\partial\mathcal{L}^n g(U_s,Y_s)}{\partial y}\gamma(Y_s)dW_s^{(2)},$$

are true martingales. For this, it suffices that

$$\mathbb{E}\int_0^t\left(\frac{\partial\mathcal{L}^n g}{\partial u}\sigma\right)^2ds<\infty,\quad\text{and}\quad\mathbb{E}\int_0^t\left(\frac{\partial\mathcal{L}^n g}{\partial y}\gamma\right)^2ds<\infty,\quad\forall t\ge 0.$$

Let us recall that since α and γ belong to $\mathcal{C}_l^0$, we have that

$$\sup_{s\le t}\mathbb{E}|Y_s|^{2m}<\infty, \tag{3.7}$$

for any $t\ge 0$ and $m\ge 1$ (this is similar to [25, Problem 5.3.15]). Hence, given the representation of Proposition 3.2, it suffices to show that for some constants $\mathcal{M}_i^n<\infty$ and non-negative integers r_i^n:

$$\left|(B_j^n)^{(i)}(y)\right|\le\mathcal{M}_i^n(1+|y|)^{r_i^n}, \tag{3.8}$$

for any $i,n\ge 0$ and $0\le j\le n$. This claim can again be shown by induction since, given that it holds true for $n-1$ and using the iterative representation for B_j^n in Proposition 3.2,

$$\left|(B_j^n)^{(i)}(y)\right|\le\sum_{\ell=0}^{i}\binom{i}{\ell}\left|\frac{1}{2}\left(\gamma^2\right)^{(\ell)}\left(B_j^{n-1}\right)^{(i-\ell+2)}\right.$$
$$\left.+(\alpha)^{(\ell)}\left(B_j^{n-1}\right)^{(i-\ell+1)}+\frac{1}{2}\left(\sigma^2\right)^{(\ell)}\left(B_{j-1}^{n-1}\right)^{(i-\ell)}\right|,$$

which can be bounded by $\mathcal{M}_i^n(1+|y|)^{r_i^n}$ since, by assumption, σ^2, α, and γ^2 belong to $\mathcal{C}_l^k$, for all $k\ge 0$. □

We remark that the previous result covers the Heston model:

$$dU_t=-\frac{1}{2}Y_t dt+\sqrt{Y_t}dW_t^{(1)},\quad dY_t=\kappa(\theta-Y_t)dt+v\sqrt{Y_t}dW_t^{(2)}. \tag{3.9}$$

Let us now use Corollary 3.3 to obtain a second order expansion for $\mathbb{E}h(U_t)$. Omitting, for the ease of notation, the evaluation of the functions B_j^k at y_0, we can write

$$\mathbb{E}h(U_t) = h(0) + B_1^1 \mathcal{L}_u h(0) t + \left\{ B_1^2 \mathcal{L}_u h(0) + B_2^2 \mathcal{L}_u^2 h(0) \right\} t^2 + O(t^3),$$

where

$$B_1^1 = \frac{1}{2}\sigma_0^2, \quad B_1^2 = \gamma_0^2 \sigma_0 \sigma_0'' + \gamma_0^2 (\sigma_0')^2 + 2\alpha_0 \sigma_0 \sigma_0', \quad B_2^2 = \frac{1}{4}\sigma_0^4. \tag{3.10}$$

Above, we set $\sigma_0 = \sigma(y_0)$, $\sigma_0' = \sigma'(y_0)$, and $\sigma_0'' = \sigma''(y_0)$, with similar notation for the other functions. A general (formal) formula for polynomial expansion of transition distributions will be as follows:

Theorem 3.4. *Under the notations and conditions of Theorem 3.1 and Corollary 3.3,*

$$\mathbb{P}(Z_t \geq z) = e^{-\lambda_\varepsilon t} \sum_{j=1}^{n} \widehat{a}_j(z) \frac{t^j}{j!} + O_{\varepsilon, z_0}(t^{n+1}), \tag{3.11}$$

where

$$\widehat{a}_j(z) := \sum_{p+q+r=j} \binom{j}{p,q,r} L_\varepsilon^q \left(\sum_{m=0}^{r} B_m^r(y_0) \mathcal{L}_u^m (\widehat{f}_{p,z}) \right)(0),$$

where we set $\widehat{f}_{0,z}(y) \equiv 0$ and where the summation is over all non-negative integers p, q, r.

Proof. It is enough to plug the expansion (3.6) into the coefficients of the first summation in (B-5) (See the proof of Theorem 3.1 in Appendix B) and rearrange terms using Proposition 3.2. Note that the last integral in (3.6) is bounded for $\widetilde{f}_{k,z} \in C_b^\infty(\mathbb{R})$ because of the representation in Proposition 3.2 and the estimates (3.7)-(3.8). □

As a way to illustrate the expansions, note that in the case of constant volatility ($\alpha(y) = \gamma(y) \equiv 0$),

$$B_k^k(y) \equiv \left(\frac{\sigma_0^2}{2} \right)^k, \quad B_j^k(y) \equiv 0, \quad \forall j \neq k, \quad k \geq 0.$$

Hence,

$$\widehat{a}_j(z) := \sum_{p+q+r=j} \binom{j}{p,q,r} \left(\frac{\sigma_0^2}{2} \right)^r L_\varepsilon^q \left(\mathcal{L}_u^r (\widehat{f}_{p,z}) \right)(0).$$

3.2. **Expansions for the call option price.**

For $z \neq 0$ and $t > 0$, let

$$G_t(z) := \mathbb{E}\left(e^{z+Z_t} - 1\right)_+, \tag{3.12}$$

where Z is the jump-diffusion process given by (1.2) and (1.3). We proceed to derive the small-time expansion of G_t as $t \downarrow 0$. We first consider the out-of-the-money case $z < 0$ from which one can easily derive the in-the-money case $z > 0$ via put-call parity (see Corollary 3.8 below). Throughout this section, we set

$$f(u) = f_z(u) := (e^{z+u} - 1)_+,$$

and we also assume the following uniform boundedness condition: there exists $0 < M < \infty$, such that

$$0 < \sigma(y) \leq M. \tag{3.13}$$

Remark 3.5. Under the uniform boundedness condition (3.13), it is easy to see that $\mathbb{E}e^{cU_t} < \infty$, for some $c > 2$. Then, a proof similar to that leading to Corollary 3.3, using the representation of Proposition 3.2, shows that (3.6) is satisfied for $g(u, y) := h(u)$, whenever $h \in C^{2n+2}$ is a subexponential function satisfying (2.8).

The next theorem gives an expansion for the out-of-the-money call option prices (its proof is given in Appendix C):

Theorem 3.6. *Let $z_0 < 0$, $n \geq 1$, and $0 < \varepsilon < -z_0/(n+1) \wedge 1$. Let the dynamics of Z be given by (1.2) and (1.3), and the conditions of both Theorem 3.1 and Corollary 3.3 as well as (3.13) be satisfied. Then there exists a $t_0 > 0$ such that, for any $0 < t < t_0$ and $z < z_0$,*

$$G_t(z) = e^{-\lambda_\varepsilon t} \sum_{j=1}^{n} \widehat{b}_j(z) \frac{t^j}{j!} + O_{\varepsilon, z_0}(t^{n+1}), \tag{3.14}$$

where

$$\widehat{b}_j(z) := \sum_{p+q+r=j} \binom{j}{p, q, r} L_\varepsilon^q \left(\sum_{m=0}^{r} B_m^r(y_0) \mathcal{L}_u^m(\widehat{f}_{p,z}) \right)(0)$$

with $\widehat{f}_{0,z}(y) \equiv 0$, and

$$\widehat{f}_{k,z}(y) := \int_{\mathbb{R}} f_z(y+u) \bar{s}_\varepsilon^{*k}(u) du = \int_{\mathbb{R}} \left(e^{z+y+u} - 1\right)_+ \bar{s}_\varepsilon^{*k}(u) du.$$

Remark 3.7. By expanding $e^{-\lambda_\varepsilon t}$ in (3.14), one obtains the coefficients in (1.6):

$$b_k(z) := \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} \widehat{b}_j(z) (-\lambda_\varepsilon)^{k-j}. \tag{3.15}$$

To deal with the in-the-money case $z > 0$, note that

$$\begin{aligned} \mathbb{E}\left(e^{z+Z_t} - 1\right)_+ &= \mathbb{E}(e^{z+Z_t} - 1) + (e^{z+Z_t} - 1)_- \\ &= e^z - 1 + \mathbb{E}(e^{z+Z_t} - 1)_-. \end{aligned}$$

The expansion of $\mathbb{E}(e^{z+Z_t} - 1)_-$ when $z > 0$ is similar to that of $\mathbb{E}\left(e^{z+Z_t} - 1\right)_+$ when $z < 0$. Therefore:

Corollary 3.8. *Let $z_0 > 0$, $n \geq 1$, and $0 < \varepsilon < z_0/(n+1) \wedge 1$. Under conditions of Theorem 3.1, there exists a $t_0 > 0$ such that, for any $0 < t < t_0$, $z > z_0$,*

$$G_t(z) = e^z - 1 + e^{\lambda_\varepsilon t} \sum_{m=1}^{n} \widetilde{b}_m(z) \frac{t^m}{m!} + O_{\varepsilon, z_0}(t^{n+1}), \tag{3.16}$$

where

$$\widetilde{b}_j(z) := \sum_{i+j+k=m} \binom{m}{i, j, k} L_\varepsilon^i \left(\sum_{l=0}^{i} B_l^i(y_0) \mathcal{L}_u^l \widehat{g}_{k,z} \right)(0)$$

with

$$\widehat{g}_{k,z}(y) := \int_{\mathbb{R}} \left(e^{y+z+u} - 1\right)_- \bar{s}_\varepsilon^{*k}(u) du.$$

3.3. **Other payoff functions.**

One of the advantages of our approach is that it can be applied to more general payoff functions. Concretely, consider a function of the form:

$$f_z(u) := \varphi(u)\mathbf{1}_{\{u\geq z\}},$$

where $\varphi \in C_b^{\infty}$. One can easily verify that, under the conditions of Theorem 3.4, we have the following expansion for $z > 0$:

$$\mathbb{E} f_z(Z_t) = e^{-\lambda_\varepsilon t}\sum_{j=1}^{n}\widetilde{a}_j(z)\frac{t^j}{j!} + O_{\varepsilon,z_0}(t^{n+1}), \tag{3.17}$$

where

$$\widetilde{a}_j(z) := \sum_{p+q+r=j}\binom{j}{p,q,r}L_\varepsilon^q\left(\sum_{m=0}^{r}B_m^r(y_0)\mathcal{L}_u^m(\widehat{f}_{p,z})\right)(0),$$

with $\widehat{f}_{0,z}(y) = 0$, and $\widehat{f}_{k,z}(y) := \int_{\mathbb{R}} f_z(y+u)\bar{s}_\varepsilon^{*k}(u)du = \int_{z-y}^{\infty}\varphi(y+u)\bar{s}_\varepsilon^{*k}(u)du$. Indeed, from the proof of Theorem 3.1 (which is the key for Theorem 3.4), the only step that requires some extra care is to justify that

$$\widetilde{f}_{k,z}(y) := \lambda_\varepsilon^{-k}\int_{z-y}^{\infty}\varphi(y+u)\bar{s}_\varepsilon^{*k}(u)du,$$

is C^∞ and $\sup_y |\widetilde{f}_{k,z}^{(j)}(y)| < \infty$. This is proved by checking (using induction) that

$$\widetilde{f}_{k,z}^{(j)}(y) = \lambda_\varepsilon^{-k}\int_{z-y}^{\infty}\varphi^{(j)}(y+u)\bar{s}_\varepsilon^{*k}(u)du + \lambda_\varepsilon^{-k}\sum_{i=0}^{j-1}(-1)^i\varphi^{(j-1-i)}(z)\bar{s}_\varepsilon^{*(k-1)} * \bar{s}_\varepsilon^{(i)}(z-y).$$

Similarly, under the stronger conditions of Theorem 3.6, one can easily consider payoff functions of the form

$$f_z(u) := \varphi(u)\mathbf{1}_{\{u\geq -z\}}, \quad (z<0),$$

with $\varphi \in C^\infty$ such that $|\varphi^{(j)}(u)| \leq M_j e^u$ for some constant $M_j < \infty$ and all $j \geq 0$.

3.4. **On the small-time large deviation principle for diffusions.**

Large deviation results of the form (3.3) have recently been developed for different stochastic volatility (SV) models. For instance, for uncorrelated SV models, Forde and Jacquier [17] shows (3.3) under the following conditions:

(3.18) The function α is bounded and uniformly Lipschitz continuous.

(3.19) $\exists\, M_2 > M_1 > 0,\ s.t.\ 0 \leq M_1 \leq \sigma(y)\wedge\gamma(y) \leq \sigma(y)\vee\gamma(y) \leq M_2 < \infty.$

(3.20) $\sigma, \gamma \in C^\infty,$ and $\sigma(y)\to\sigma_\pm,\ \gamma(y)\to\gamma_\pm,$ as $y\to\pm\infty.$

(3.21) σ and γ are diffeomorphisms with $\sigma' > 0$ and $\gamma' > 0$.

(3.22) $\exists\, y_c \in \mathbb{R},$ such that $\sigma'' > 0,\ \gamma'' > 0\ for\ y < y_c,\ \sigma'' < 0,\ \gamma'' < 0$ for $y > y_c$ and $\sigma'\vee\gamma' < M < \infty$ for some $M > 0$.

(3.23) The function $u \mapsto \dfrac{\gamma(\sigma^{-1}(u))}{u}$ is non-increasing.

We refer to [17] for an explicit expression for the rate function I, which is not relevant here. The Heston model (3.9) (even with correlated Wiener processes $W^{(1)}$ and $W^{(2)}$) was also considered in [16] and [18].

4. Expansions for the call-option price under exponential Lévy Models

In this section, we point out the expansion of the call-option price under an exponential Lévy model, which is a particular case of the jump-diffusion models (1.2) and (1.3). More precisely, let the log-return process Z be a general Lévy process with the generating triplet (b, σ^2, ν). Then, the following expansion for the out-of-money call option price holds true (see also Figueroa-López and Forde [14]). Note that the process Z in this section is the same as the process X in Section 2 and hence all the notations are transferred accordingly.

Corollary 4.1. *Let $z_0 < 0$, $n \geq 1$, and $0 < \varepsilon < -z_0/(n+1) \wedge 1$. Let $Z = (Z_t)_{t \geq 0}$ be a Lévy process with triplet (b, σ^2, ν) satisfying (2.4)-(2.5). Then there exists $t_0 > 0$ such that, for any $z < z_0$ and $0 < t < t_0$,*

$$G_t(z) = e^{-\lambda_\varepsilon t} \sum_{j=1}^{n} c_j(z) \frac{t^j}{j!} + O_{\varepsilon, z_0}(t^{n+1}), \tag{4.1}$$

where

$$c_j(z) := \sum_{k=1}^{j} \binom{j}{k} L_\varepsilon^{j-k} \widehat{h}_{k,z}(0),$$

with

$$\widehat{h}_{k,z}(y) := \int_{\mathbb{R}} \left(e^{z+y+u} - 1\right)_+ \bar{s}_\varepsilon^{*k}(u) du.$$

For the in-the-money case, similarly to Corollary (3.8), we also have

Corollary 4.2. *Let $z_0 > 0$, $n \geq 1$, and $0 < \varepsilon < z_0/(n+1) \wedge 1$. Then, there exists $t_0 > 0$ such that, for any $z > z_0$ and $0 < t < t_0$,*

$$G_t(z) = e^z - 1 + e^{-\lambda_\varepsilon t} \sum_{j=1}^{n} \tilde{c}_j(z) \frac{t^j}{j!} + O_{\varepsilon, z_0}(t^{n+1}), \tag{4.2}$$

where

$$\tilde{c}_j(z) := \sum_{k=1}^{j} \binom{j}{k} L_\varepsilon^{j-k} \tilde{h}_{k,z}(0), \quad \tilde{h}_{k,z}(y) := \int_{\mathbb{R}} \left(e^{z+y+u} - 1\right)_- \bar{s}_\varepsilon^{*k}(u) du.$$

Given that in (3.10) $B_0^0(y_0)$ and $B_1^1(y_0)$ depend only on $\sigma(y_0)$, it is interesting to note that the first two coefficients in our expansions (3.11) and (3.14) coincide with the coefficients corresponding to an exponential Lévy model with variance $\sigma^2 = \sigma^2(y_0)$. In fact, the initial values of α and γ begin to appear with the coefficient $\widehat{a}_3(z)$ and $\widehat{b}_3(z)$ through the coefficients B_1^2 and B_2^2 in (3.10).

Here are the first two coefficients of (3.11) for $\varepsilon > 0$ small enough:

$$\widehat{a}_1(z) = B_0^0(y_0)\widehat{f}_{1,z}(0) = \int_{\mathbb{R}} f_z(u)\bar{s}_\varepsilon(u)du = \int_z^\infty s(u)du;$$

$$\begin{aligned}\widehat{a}_2(z) &= \underbrace{2L_\varepsilon(\widehat{f}_{1,z})(0)}_{p=1,q=1,r=0} + \underbrace{2B_1^1(y_0)\mathcal{L}_u(\widehat{f}_{1,z})(0)}_{p=1,q=0,r=1} + \underbrace{\widehat{f}_{2,z}(0)}_{p=2,q=0,r=0} \\ &= 2\left(b_\varepsilon s(z) - \int_{\mathbb{R}}\int_0^1 s'(z-\beta u)(1-\beta)d\beta u^2 s_\varepsilon(u)du\right) \\ &\quad - \sigma^2(y_0)(s'(z)+s(z)) + \int_{\mathbb{R}^2} \mathbf{1}_{\{u_1+u_2\geq z\}}\bar{s}_\varepsilon(u_1)\bar{s}_\varepsilon(u_2)du_1du_2.\end{aligned}$$

The corresponding coefficients for (3.14) are obtained as above with $f_z(x) = \mathbf{1}_{\{x\geq z\}}$ replaced by $f_z(y) := (e^{z+y}-1)_+$ with $z<0$. Hence, for $\varepsilon > 0$ small enough,

$$\widehat{b}_1(z) = \int_{\mathbb{R}} \left(e^{z+u}-1\right)_+ s(u)du;$$

$$\begin{aligned}\widehat{b}_2(z) &= \sigma^2(y_0)s(-z) + 2b_\varepsilon \int_{-z}^\infty e^{z+u}s(u)du + \int_{\mathbb{R}^2}\left(e^{z+u_1+u_2}-1\right)_+ \bar{s}_\varepsilon(u_1)\bar{s}_\varepsilon(u_2)du_1du_2 \\ &\quad + 2\int_{\mathbb{R}}\int_0^1 (1-\beta)\left(\int_{-z-\beta u}^\infty e^{z-\beta u+w}s(w)dw + s(-z-\beta u)\right)d\beta u^2 s_\varepsilon(u)du.\end{aligned}$$

In the previous expressions one can substitute $c_\varepsilon(y)$ and $\bar{c}_\varepsilon(y)$ by $\mathbf{1}_{0<|y|<\varepsilon}$ and $\mathbf{1}_{|y|\geq\varepsilon}$, respectively.

Combining (1.6), (1.7), and the expression for $\hat{b}_1(z) = b_1(z)$ above, we obtain the expansion for the price function of the out-of-money call option near the expiration T:

$$\begin{aligned}C(t,s) &= Ke^{-r(T-t)}G_{T-t}(\ln s - \ln K) \\ &= (T-t)\int_{\mathbb{R}} (se^u - K)_+ s(u)du + O_{\varepsilon,\ln(s/K)}\big((T-t)^2\big). \qquad (4.3)\end{aligned}$$

5. Asymptotics of the implied volatility

Using the leading term of the time-t price for the out-of-money call option as computed in the previous section, we now obtain the asymptotic behavior of the implied volatility $\hat{\sigma}(t;s)$ near T. It is defined implicitly by the equation

$$C(t,s) = C_{BS}(t,s;\hat{\sigma}(t;s),r), \qquad (5.1)$$

where $C_{BS}(t,s;\sigma,r)$ is the classical time-t Black-Scholes call-option price corresponding to an interest rate r, a volatility σ, and time t spot price s. We shall need the following well-known result (see, e.g., Lemma 2.5 in [20]):

Lemma 5.1. *Let $C_{BS}(t,s;\sigma,r)$ be the classical Black-Scholes call price function. Then, as $t \uparrow T$,*

$$\begin{aligned}C_{BS}(t,s;\sigma,r) \;&\sim\; \frac{1}{\sqrt{2\pi}}\frac{K\sigma^3(T-t)^{3/2}}{(\ln K - \ln s)^2}\exp\left[-\frac{(\ln K-\ln s)^2}{2\sigma^2(T-t)}\right] \\ &\quad\times \exp\left[-\frac{\ln K - \ln s}{2} + \frac{r(\ln K - \ln s)}{\sigma^2}\right] + R(t,s;\sigma,r). \qquad (5.2)\end{aligned}$$

The remainder satisfies

$$|R(t,s;\sigma,r)| \leq M(T-t)^{5/2} \exp\left[-\frac{(\ln K - \ln s)^2}{2\sigma^2(T-t)}\right], \tag{5.3}$$

where $M = M(s,\sigma,r,K)$ is uniformly bounded if all the indicated parameters vary in a bounded region.

The next result gives the asymptotic behavior of $\hat{\sigma}(t,s)$. This has already been obtained for a pure-Lévy processes (see, e.g., [38] and [14]) and is presented here for the sake of completeness:

Proposition 5.2. *Let $\hat{\sigma}(t;s)$ be the implied volatility when the stock price (1.1) is s at time t. Then, as $t \uparrow T$,*

$$\hat{\sigma}^2(t;s) \sim \frac{(\ln K - \ln s)^2}{-2(T-t)\ln(T-t)}. \tag{5.4}$$

Proof. Using the leading terms in (4.3) and (5.2), we obtain that as $t \uparrow T$:

(5.5)

$$(T-t)u(s,K) \sim v(s,K)\hat{\sigma}^3(t;s)(T-t)^{3/2} \exp\left[-\frac{(\ln K - \ln s)^2}{2\hat{\sigma}^2(t;s)(T-t)} + \frac{r(\ln K - \ln s)}{\hat{\sigma}^2(t;s)}\right],$$

where

$$u(s,K) = \int_{\mathbb{R}} (se^u - K)_+ \, s(u) du,$$
$$v(s,K) = \frac{1}{\sqrt{2\pi}} \frac{K}{(\ln K - \ln s)^2} \exp\left[-\frac{\ln K - \ln s}{2}\right].$$

Assume that $\limsup_{t\uparrow T} \hat{\sigma}(t)(T-t)^{1/2} = c \in (0,+\infty)$, then $\limsup_{t\uparrow T} \hat{\sigma}(t) \uparrow \infty$, and thus

$$\limsup_{t\uparrow T} \left(\hat{\sigma}(t)(T-t)^{1/2}\right)^3 \exp\left[-\frac{(\ln K - \ln s)^2}{2\hat{\sigma}^2(t)(T-t)} + \frac{r(\ln K - \ln s)}{\hat{\sigma}^2(t)}\right]$$
$$= c^3 \exp\left(-\frac{(\ln K - \ln s)^2}{2c^2}\right) \neq 0.$$

So the right hand side of (5.5) does not converge to 0 while the left hand side does, which is clearly a contradiction.

Now if $\limsup_{t\uparrow T} \hat{\sigma}(t)(T-t)^{1/2} = +\infty$, then $\limsup_{t\uparrow T} \hat{\sigma}(t) \uparrow +\infty$, and thus

$$\limsup_{t\uparrow T} \left(\hat{\sigma}(t)(T-t)^{1/2}\right)^3 \exp\left[-\frac{(\ln K - \ln s)^2}{2\hat{\sigma}^2(t)(T-t)} + \frac{r(\ln K - \ln s)}{\hat{\sigma}^2(t)}\right] = +\infty.$$

Again we obtain the same contradiction. Therefore, we have $\limsup_{t\uparrow T} \hat{\sigma}(t)(T-t)^{1/2} = 0$, which obviously implies that $\lim_{t\uparrow T} \hat{\sigma}(t)(T-t)^{1/2} = 0$. Then, (5.5) can now be equivalently written as

$$\exp\left[-\frac{(\ln K - \ln s)^2}{2\hat{\sigma}^2(t)(T-t)} + \frac{r(\ln K - \ln s)}{\hat{\sigma}^2(t)} + 3\ln\left(\hat{\sigma}(t)(T-t)^{1/2}\right) - \ln(T-t)\right] \sim \frac{u(s,K)}{v(s,K)}.$$

Hence, as $t \uparrow T$,

$$\frac{1}{\big(\hat{\sigma}(t)(T-t)^{1/2}\big)^2}\Bigg[r(\ln K - \ln s)(T-t) + 3\big(\hat{\sigma}(t)(T-t)^{1/2}\big)^2 \ln\Big(\hat{\sigma}(t)(T-t)^{1/2}\Big)$$
$$-\frac{(\ln K - \ln s)^2}{2} - \big(\hat{\sigma}(t)(T-t)^{1/2}\big)^2 \ln(T-t) - \big(\hat{\sigma}(t)(T-t)^{1/2}\big)^2 \ln\frac{u(s,K)}{v(s,K)}\Bigg],$$

which converges to 0. Finally, note that

$$\begin{aligned}\lim_{t\uparrow T}\big(\hat{\sigma}(t)(T-t)^{1/2}\big)^2 \ln\Big(\hat{\sigma}(t)(T-t)^{1/2}\Big) &= 0,\\ \lim_{t\uparrow T} r(\ln K - \ln s)(T-t) &= 0,\end{aligned}$$

and therefore,

$$\lim_{t\uparrow T}\left(\frac{(\ln K - \ln s)^2}{2} + \hat{\sigma}^2(t)(T-t)\ln(T-t)\right) = 0,$$

which directly implies (5.4). □

6. Small-time expansions for the Lévy transition densities

In this part, we revisit the important problem of finding small-time expansions for the transition densities of Lévy processes. This problem has been considered in Rüschendorf and Woerner [36] and also Figueroa-López and Houdré [13]. As in Section 2.1, we consider a general Lévy process X with Lévy triplet (b, σ^2, ν). It is well-known that under general conditions (see, e.g, [27] and [33]):

$$\lim_{t\to 0}\frac{1}{t} f_t(x) = s(x), \quad (x \neq 0), \tag{6.1}$$

where f_t denotes the probability density of X_t and s is the Lévy density of ν (both densities are assumed to exist). In many applications, the following uniform convergence result is more desirable

$$\lim_{t\to 0}\sup_{|x|\geq\eta}\left|\frac{1}{t} f_t(x) - s(x)\right| = 0, \tag{6.2}$$

for a fixed $\eta > 0$. The limit (6.2) is related to the following general expansions for the transition densities:

$$f_t(x) = \sum_{k=1}^{n}\frac{a_k(x)}{k!}t^k + t^{n+1}O_\eta(1), \tag{6.3}$$

valid for any $|x| \geq \eta$ and $0 < t < t_0$, with t_0 possibly depending on the given $\eta > 0$ and $n \geq 0$. Above, $O_\eta(1)$ denotes a function of x and t such that

$$\sup_{0<t\leq t_0}\sup_{|x|\geq\eta}|O_\eta(1)| < \infty.$$

Note that (6.2) follows from (6.3) when $n = 1$ and $a_k(x) = s(x)$.

Rüschendorf and Woerner [36] were the first to propose (6.2) building on results of Léandre [27], who prove the point-wise convergence (6.1). In both papers, the standing conditions on the Lévy density s of the Lévy process X are as follows:

$$\liminf_{\eta\to 0}\eta^{\alpha-2}\int_{-\eta}^{\eta} z^2 s(z)dz>0,\ (0<\alpha<2); \tag{6.4}$$

$$s\in C^{\infty}(\mathbb{R}\setminus\{0\}); \tag{6.5}$$

$$\int_{|z|\geq\eta}\frac{|s'(z)|^2}{s(z)}dz<\infty,\ \forall\eta>0; \tag{6.6}$$

$$\exists h\in C^{\infty} \text{ such that } h(z)=O(z^2)\ (z\to 0),\ h(z)>0 \text{ if } s(z)>0, \textit{ and} \tag{6.7}$$
$$\int_{|z|\leq 1}\left|\frac{d}{dz}h(z)s(z)\right|^2\frac{1}{s(z)}dz<\infty.$$

Condition (6.4) is used to conclude the existence of a C^{∞} transition density f_t (see [37, Chapter 5]), while (6.5)-(6.7) are needed to establish an estimate for the transition density using Malliavin calculus. However, the method of proof of [36] had a gap so that one can only derive the first order expansion in (6.3) (see the introduction of [13] for more details). Recently, Figueroa-López and Houdré [13] obtained (6.3) under the following assumptions:

$$\gamma_{\eta,k}:=\sup_{|x|\geq\eta}|s^{(k)}(x)|<\infty, \text{ and} \tag{6.8}$$

$$\limsup_{t\searrow 0}\sup_{|x|\geq\eta}|f_t^{(k)}(x)|<\infty,\quad \forall k\geq 0 \text{ and } \forall\eta>0. \tag{6.9}$$

Condition (6.8) is quite mild but condition (6.9) could be hard to prove in general due to the inaccessibility of closed-form expressions for the densities f_t. Nevertheless [13] shows that condition (6.9) is satisfied by, e.g., the CGMY model of [8] (or Koponen [26]) and by other types of tempered stable Lévy processes (as defined in [34]).

In this section, we show that (6.9) is not necessary to obtain (6.3). See Appendix C for the proof of the following result:

Theorem 6.1. *Let $\eta>0$ and $n\geq 1$, and let the conditions in lines (6.4)-(6.8) be satisfied. Then, (6.3) holds true for all $0<t\leq 1$ and $|x|\geq\eta$. Moreover, there exists $\varepsilon_0(\eta,n)>0$ such that for all $0<\varepsilon<\varepsilon_0$, the coefficients a_k admit the following representation (which is moreover constant for any $0<\varepsilon<\varepsilon_0$):*

$$a_k(x):=\sum_{j=1}^{k}\binom{k}{j}(-\lambda_{\varepsilon})^{k-j}\sum_{i=1}^{j}\binom{j}{i}L_{\varepsilon}^{j-i}\hat{s}_{i,x}(0), \tag{6.10}$$

where $\hat{s}_{i,x}(u):=\bar{s}_{\varepsilon}^{*i}(x-u)$.

Remark 6.2. Combining the proofs of Theorem 3.1 and of Theorem 6.1, it is possible to obtain a small-time expansion for the jump-diffusion model (1.2)-(1.3) assuming, for instance, that the stochastic volatility model admits a density function d_t satisfying the small-time estimate:

$$\sup_{|x|\geq\eta}d_t(x)\leq M_{p,\eta}t^p,$$

for any $p\geq 1$ and $0<t<t_0(p,\eta)$ and some constant $M_{p,\eta}<\infty$.

Appendix A. Proof of Proposition 2.1

Let us show (2.7) for $n = 1$ (the other cases are easily obtained by induction). First, applying Itô's lemma ([24, Theorem I.4.56]),

$$g(X_t) = g(0) + \int_0^t Lg(X_u)du + \sigma \int_0^t g'(X_u)dW_u + \int_0^t \int_{\mathbb{R}} \left\{ g(X_{u^-} + z) - g(X_{u^-}) - g'(X_{u^-})z\mathbf{1}_{|z|\leq 1} \right\} \bar{\mu}(du, dz),$$

where one can easily check that $Lg(x)$ is well-defined from the continuity of $g^{(i)}$ and (2.8). Indeed, there exists a constant M_i such that $|g^{(i)}(x)| \leq M_i e^{\frac{c}{2}|x|}$, for all x, and thus,

$$\left| \int_{|z|\geq 1} g(x+z)\nu(dz) \right| \leq M_1 \int_{|z|\geq 1} e^{\frac{c}{2}|z|}\nu(dz)e^{\frac{c}{2}|x|},$$
$$\left| \int_{|z|\leq 1} (g(x+z) - g(x) - g'(x)z)\nu(dz) \right| \leq M_2 e^{\frac{c}{2}} \int_{|z|\leq 1} z^2 \nu(dz) e^{\frac{c}{2}|x|}.$$

Next, we show that the last two terms of the expansion of $g(X_t)$ above are true martingales. Indeed, it suffices that

$$\mathbb{E}\int_0^t |g'(X_u)|^2 du < \infty, \tag{A-1}$$

$$\mathbb{E}\int_0^t \int_{|z|>1} |g(X_u + z) - g(X_u)| \nu(dz)du < \infty, \tag{A-2}$$

$$\mathbb{E}\int_0^t \int_{|z|\leq 1} |g(X_u + z) - g(X_u) - g'(X_u)z|^2 \nu(dz)du < \infty. \tag{A-3}$$

Using (2.8) and the continuity of g', there exists a constant $M > 0$ such that

$$\mathbb{E}\int_0^t |g'(X_u)|^2 du \leq M \int_0^t \mathbb{E}e^{c|X_u|}du \leq M \int_0^t \mathbb{E}e^{cX_u}du + \int_0^t \mathbb{E}e^{-cX_u}du < \infty,$$

for any $t \geq 0$. Similarly, setting $\bar{B} = \{z : |z| > 1\}$, (A-2) is satisfied since

$$\mathbb{E}\int_0^t \int_{\bar{B}} |g(X_u + z) - g(X_u)| d\nu du \leq \mathbb{E}\int_0^t \int_{\bar{B}} \left| \int_0^z g'(X_u + w)dw \right| d\nu du \leq M \int_0^t \mathbb{E}e^{c|X_u|}du \int_{\bar{B}} \int_0^{|z|} e^{cw} dw d\nu < \infty.$$

Also, setting $B = \{z : |z| \leq 1\}$,

$$\mathbb{E}\int_0^t \int_B |g(X_u + z) - g(X_u) - g'(X_u)z|^2 \nu(dz)du$$
$$\leq \mathbb{E}\int_0^t \int_B \int_0^1 |g''(X_u + z\beta)|^2 (1-\beta)^2 d\beta z^4 d\nu du$$
$$\leq \int_0^t \mathbb{E}e^{c|X_u|}du \int_B \int_0^1 e^{c|z|\beta}(1-\beta)^2 d\beta z^4 d\nu < \infty.$$

We then have that

$$\mathbb{E}g(X_t) = g(0) + \mathbb{E}\int_0^t Lg(X_u)du,$$

which leads to (2.6) provided $\int_0^t \mathbb{E}\,|Lg(X_u)|\,du < \infty$. The later is proved using (2.8) and similar arguments.

In order to obtain (2.7) for $n = 1$ by iterating (2.6), we need to show that for any C^4 function g satisfying (2.8),

$$\limsup_{|y|\to\infty} e^{-\frac{c}{2}|y|}|(Lg)^{(i)}(y)| < \infty, \tag{A-4}$$

for $i = 0, 1, 2$. To this end, we first note that

$$(Lg)^{(i)}(y) = bg^{(i+1)}(y) + \frac{\sigma^2}{2}g^{(i+2)}(y) + \int_{\mathbb{R}}(g^{(i)}(y+z) - g^{(i)}(y) - zg^{(i+1)}(y)\mathbf{1}_{|z|\le 1})\nu(dz)$$

for $i = 0, 1, 2$. Hence, it is sufficient to show (A-4) when $i = 0$, and we have

$$e^{-\frac{c}{2}|y|}|Lg(y)| \le be^{-\frac{c}{2}|y|}|g'(y)| + \frac{\sigma^2}{2}e^{-\frac{c}{2}|y|}|g''(y)| \tag{A-5}$$

$$+ e^{-\frac{c}{2}|y|}\int_{|z|>1}|g(y+z) - g(y)|\nu(dz) \tag{A-6}$$

$$+ e^{-\frac{c}{2}|y|}\int_{|z|\le 1}|g(y+z) - g(y) - zg'(y)|\nu(dz). \tag{A-7}$$

The limits of the right-hand terms in (A-5) as $|y| \to \infty$ are trivially finite by the assumption (2.8). For the term in (A-6), again by the assumption (2.8) and the continuity of $g^{(i)}$, there exists $M > 0$ such that,

$$|g^{(i)}(y)| \le Me^{\frac{c}{2}|y|}, \quad i = 0, 1, 2.$$

It follows that

$$\begin{aligned} e^{-\frac{c}{2}|y|}\int_{|z|>1}|g(y+z) - g(y)|\nu(dz) &= e^{-\frac{c}{2}|y|}\int_{|z|>1}\Big|\int_0^z g'(y+w)\,dw\Big|\nu(dz) \\ &\le M\int_{|z|>1}\Big(\int_0^{|z|} e^{\frac{c}{2}w}dw\Big)\nu(dz) \\ &= M\int_{|z|>1} e^{\frac{c}{2}|z|}\nu(dz) < \infty, \end{aligned}$$

which immediately implies that

$$\limsup_{|y|\to\infty} e^{-\frac{c}{2}|y|}\int_{|z|>1}|g(y+z) - g(y)|\nu(dz) < \infty. \tag{A-8}$$

Similarly, we can show that the limit of (A-7) as $|y| \to \infty$ is finite. Therefore, we can iterate (2.6) to obtain (2.7) for $n = 1$. $\square$

Appendix B. Proof of Theorem 3.1

We will analyze each term on the right-hand side of the expansion of $\mathbb{E}f(Z_t)$ given in (3.1)-(3.2):

(1) For any $z \geq z_0$, we have

$$\mathbb{E}f_z\left(U_t + X_t^\varepsilon\right) = \mathbb{P}(U_t + X_t^\varepsilon \geq z) \leq \mathbb{P}(U_t \geq z/2) + \mathbb{P}(X_t^\varepsilon \geq z/2). \tag{B-1}$$

By our assumption (3.3), there exists $t_0(z_0) > 0$ such that for any $0 < t \leq t_0$, $z \geq z_0 > 0$,

$$\mathbb{P}(U_t \geq z/2) \leq \mathbb{P}(U_t \geq z_0/2) \leq \exp\left(-\frac{d(z_0/2)^2}{4t}\right), \tag{B-2}$$

which can be seen to be $O_{z_0}(t^{n+1})$. Also, the second term on the right-hand-side of (B-1) is $O_{\varepsilon,z_0}(t^{n+1})$ in light of (2.3) by taking $a := (n+1)/z_0$ and using that $0 < \varepsilon < z_0/(n+1) \wedge 1$.

(2) The second term in (3.1) is also $O_{\varepsilon,z_0}(t^{n+1})$ because $f \leq 1$ and clearly $e^{-\lambda_\varepsilon t}\sum_{k=n+1}^{\infty}(\lambda_\varepsilon t)^k/k! \leq (\lambda_\varepsilon t)^{n+1} = O(t^{n+1})$.

(3) We proceed to work out those terms in (3.2). Using the independence of U and X, we have

$$\mathbb{E}f_z\left(U_t + X_t^\varepsilon + \sum_{i=1}^{k}\xi_i\right) = \mathbb{E}\widetilde{f}_{k,z}\left(U_t + X_t^\varepsilon\right) = \mathbb{E}\breve{f}_{k,z,t}(X_t^\varepsilon), \tag{B-3}$$

where

$$\widetilde{f}_{k,z}(y) := (\lambda_\varepsilon)^{-k}\int_{z-y}^{\infty}\bar{s}_\varepsilon^{*k}(u)du \quad \text{and} \quad \breve{f}_{k,z,t}(y) := \mathbb{E}\widetilde{f}_{k,z}\left(U_t + y\right).$$

In particular, by the assumption (2.4),

$$\widetilde{f}_{k,z}^{(j)}(y) = (\lambda_\varepsilon)^{-k}(-1)^{j-1}\bar{s}_\varepsilon^{*(k-1)} * \bar{s}_\varepsilon^{(j-1)}(z-y),$$
$$\sup_{y,z}\left|\widetilde{f}_{k,z}^{(j)}(y)\right| \leq \lambda_\varepsilon^{-1}\|\bar{s}_\varepsilon^{(j-1)}\|_\infty \leq \lambda_\varepsilon^{-1}\max_{0\leq i\leq j-1}\gamma_{i,\varepsilon/2} := \Gamma_\varepsilon < \infty.$$

It follows that $\breve{f}_{k,z,t} \in C_b^\infty(\mathbb{R})$ and moreover,

$$\breve{f}_{k,z,t}^{(j)}(y) = \mathbb{E}\widetilde{f}_{k,z}^{(j)}\left(U_t + y\right), \text{ and } \sup_{z,y}\left|\breve{f}_{k,z,t}^{(j)}(y)\right| \leq \Gamma_\varepsilon, \text{ for any } j \geq 0. \tag{B-4}$$

We will then be able to apply the iterated formula (2.7) to get

$$\mathbb{E}\breve{f}_{k,z,t}(X_t^\varepsilon) = \sum_{i=0}^{n-k}\frac{t^i}{i!}L_\varepsilon^i\breve{f}_{k,z,t}(0) + \frac{t^{n-k+1}}{(n-k)!}\int_0^1(1-\alpha)^{n-k}\mathbb{E}\{L_\varepsilon^{n-k+1}\breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)\}d\alpha. \tag{B-5}$$

It follows from the representation in Lemma 2.2 and (B-4) that

$$\sup_z\int_0^1(1-\alpha)^{n-k}\mathbb{E}\left(L_\varepsilon^{n-k+1}\breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)\right)d\alpha < \infty,$$

and thus the second term on the right hand side of (B-5) is $O_{\varepsilon,z_0}(t^{n-k+1})$.

(4) Combining (3.1), (3.2) and (B-5), we obtain

$$\begin{aligned}\mathbb{E}f(Z_t) &= e^{-\lambda_\varepsilon t}\sum_{k=1}^{n}\frac{(\lambda_\varepsilon t)^k}{k!}\mathbb{E}\breve{f}_{k,z,t}(X_t^\varepsilon) + O_{\varepsilon,z_0}(t^{n+1})\\ &= e^{-\lambda_\varepsilon t}\sum_{k=1}^{n}\frac{(\lambda_\varepsilon t)^k}{k!}\sum_{i=0}^{n-k}\frac{t^i}{i!}L_\varepsilon^i\breve{f}_{k,z,t}(0) + O_{\varepsilon,z_0}(t^{n+1})\\ &= e^{-\lambda_\varepsilon t}\sum_{j=1}^{n}\frac{t^j}{j!}\sum_{k=1}^{j}\binom{j}{k}\lambda_\varepsilon^k L_\varepsilon^{j-k}\breve{f}_{k,z,t}(0) + O_{\varepsilon,z_0}(t^{n+1}).\end{aligned}$$

Using again the representation in Lemma 2.2 and (B-4), it follows that

$$L_\varepsilon^{j-k}\breve{f}_{k,z,t}(x) = L_\varepsilon^k\left[\mathbb{E}\widetilde{f}_{k,z}(U_t+\cdot)\right](x) = \lambda_\varepsilon^{-k}L_\varepsilon^k\left[\mathbb{E}\widehat{f}_{k,z}(U_t+\cdot)\right](x),$$

and (3.4) follows. □

Appendix C. Proof of Theorem 3.6

We will analyze each term in (3.1) and (3.2).

(1) For $z \le z_0 < 0$,

$$\begin{aligned}\text{(C-1)}\quad \mathbb{E}f_z(U_t+X_t^\varepsilon) &= \mathbb{E}\left(e^{z+U_t+X_t^\varepsilon}-1\right)_+ \le \mathbb{E}\left(e^{U_t+X_t^\varepsilon}\mathbf{1}_{\{U_t+X_t^\varepsilon\ge -z\}}\right)\\ &\le \left(\mathbb{E}e^{2U_t+2X_t^\varepsilon}\mathbb{P}(U_t+X_t^\varepsilon\ge -z)\right)^{1/2}\\ &\le \left(\mathbb{E}e^{2U_t}\mathbb{E}e^{2X_t^\varepsilon}\right)^{1/2}\left(\mathbb{P}(U_t\ge -z/2)+\mathbb{P}(X_t^\varepsilon\ge -z/2)\right)^{1/2},\\ &= e^{t\Psi(2)/2}\left(\mathbb{E}e^{2U_t}\right)^{1/2}\left(\mathbb{P}(U_t\ge -z/2)+\mathbb{P}(X_t^\varepsilon\ge -z/2)\right)^{1/2},\end{aligned}$$

where Ψ is the characteristic exponent of X^ε. Since $M_t := e^{U_t}$ satisfies the SDE $dM_t = M_t\sigma(Y_t)dW_t^{(1)}$ and using the Davis-Burkhölder-Gundy inequality,

$$\begin{aligned}\mathbb{E}e^{2U_t} &= \mathbb{E}\left(1+\int_0^t M_s\sigma(Y_s)dW_s^{(1)}\right)^2\\ &\le 2+2\mathbb{E}\left(\int_0^t e^{U_s}\sigma(Y_s)dW_s^{(1)}\right)^2 \le 2+2M^2\,\mathbb{E}\int_0^t e^{2U_s}ds.\end{aligned}$$

By Gronwall's Inequality,

$$\mathbb{E}e^{2U_t} \le 2e^{2M^2 t} = O_{\varepsilon,z_0}(1).$$

Therefore, the right-hand-side of (C-1) can be made $o_{\varepsilon,z_0}(t^{n+1})$ by (2.3) and (3.3).

(2) The second summation in (3.1) is also $O_{\varepsilon,z_0}(t^{n+1})$ since for any $k \ge n+1$,

$$\begin{aligned}\mathbb{E}f_z(U_t+X_t^\varepsilon+\sum_{i=1}^{k}\xi_i) &\le e^z\mathbb{E}e^{U_t}\mathbb{E}e^{X_t^\varepsilon}(\mathbb{E}e^{\xi_1})^k\\ &\le \lambda_\varepsilon^{-k}e^{t\Psi(1)}(\int_{\mathbb{R}}e^x\bar{s}_\varepsilon(x)dx)^k.\end{aligned}$$

(3) To work out the summation in (3.2), recall that by the independence of U and X, for any $1 \leq k \leq n$,

$$\mathbb{E} f_z \left(U_t + X_t^\varepsilon + \sum_{i=1}^{k} \xi_i \right) = \mathbb{E} \widetilde{f}_{k,z} \left(U_t + X_t^\varepsilon \right) = \mathbb{E} \breve{f}_{k,z,t}(X_t^\varepsilon),$$

where

$$\breve{f}_{k,z,t}(x) = \mathbb{E} \widetilde{f}_{k,z} \left(U_t + x \right) \quad \text{and} \quad \widetilde{f}_{k,z}(x) = \mathbb{E} f_z \left(x + \sum_{i=1}^{k} \xi_i \right).$$

Let us show that $\tilde{f}_{k,z}$ is C^∞. Indeed, since

$$\widetilde{f}_{k,z}(x) = \lambda_\varepsilon^{-k} \int_{\mathbb{R}^{k-1}} \int_{-\sum_{\ell=2}^{k} u_\ell - z - x}^{\infty} \left(e^{z+x+\sum_{\ell=1}^{k} u_\ell} - 1 \right) \bar{s}_\varepsilon(u_1) du_1 \prod_{\ell=2}^{k} \bar{s}_\varepsilon(u_\ell) du_\ell,$$

and $\bar{s}_\varepsilon \in C_b^\infty$, we have that

$$\begin{aligned}
\widetilde{f}'_{k,z}(x) &= \lambda_\varepsilon^{-k} \int_{\mathbb{R}^{k-1}} \int_{-\sum_{\ell=2}^{k} u_\ell - z - x}^{\infty} e^{z+x+\sum_{\ell=1}^{k} u_\ell} \bar{s}_\varepsilon(u_1) du_1 \prod_{\ell=2}^{k} \bar{s}_\varepsilon(u_\ell) du_\ell, \\
\widetilde{f}''_{k,z}(x) &= \lambda_\varepsilon^{-k} \int_{\mathbb{R}^{k-1}} \int_{-\sum_{\ell=2}^{k} u_\ell - z - x}^{\infty} e^{z+x+\sum_{\ell=1}^{k} u_\ell} \bar{s}_\varepsilon(u_1) du_1 \prod_{\ell=2}^{k} \bar{s}_\varepsilon(u_\ell) du_\ell \\
&\quad + \lambda_\varepsilon^{-k} \int_{\mathbb{R}^{k-1}} \bar{s}_\varepsilon \left(-\sum_{\ell=2}^{k} u_\ell - z - x \right) \prod_{\ell=2}^{k} \bar{s}_\varepsilon(u_\ell) du_\ell.
\end{aligned}$$

Using induction, we see that

$$\begin{aligned}
\text{(C-2)} \qquad \widetilde{f}^{(i)}_{k,z}(x) &= \lambda_\varepsilon^{-k} \int_{\mathbb{R}} \int_{-\sum_{\ell=2}^{k} u_\ell - z - x}^{\infty} e^{z+x+\sum_{\ell=1}^{k} u_\ell} \bar{s}_\varepsilon(u_1) du_1 \prod_{\ell=2}^{k} \bar{s}_\varepsilon(u_\ell) du_\ell \\
&\quad + \lambda_\varepsilon^{-k} \sum_{j=0}^{i-2} (-1)^j \int_{\mathbb{R}^{k-1}} \bar{s}_\varepsilon^{(j)} \left(-\sum_{\ell=2}^{k} u_\ell - z - x \right) \prod_{\ell=2}^{k} \bar{s}_\varepsilon(u_\ell) du_\ell.
\end{aligned}$$

In view of (2.4), there exists a constant $M_{i,\varepsilon} < \infty$ such that, for any $i \geq 1$,

$$\begin{aligned}
\text{(C-3)} \qquad \left| \widetilde{f}^{(i)}_{k,z}(U_t + x) \right| &\leq \lambda_\varepsilon^{-k} \int_{\mathbb{R}^k} e^{z+x+\sum_{\ell=1}^{k} u_\ell} \prod_{\ell=1}^{k} \bar{s}_\varepsilon(u_\ell) du_\ell \cdot e^{U_t} \\
&\quad + M_{i,\varepsilon} \lambda_\varepsilon^{-k} \sum_{j=0}^{i-2} \int_{\mathbb{R}^{k-1}} \prod_{\ell=2}^{k} \bar{s}_\varepsilon(u_\ell) du_\ell \cdot \max_{0 \leq j \leq i} \gamma_{j,\varepsilon/2}.
\end{aligned}$$

The right-hand side of (C-3) is integrable because $\mathbb{E} e^{U_t} = 1$. By dominated convergence, we conclude that $\breve{f}_{k,z,t} \in C^\infty(\mathbb{R})$, and also,

$$\breve{f}^{(i)}_{k,z,t}(x) = \mathbb{E} \left[\widetilde{f}^{(i)}_{k,z}(U_t + x) \right], \ \forall i \geq 0, \quad \text{and} \quad \limsup_{|x| \to \infty} e^{-\frac{c}{2}|x|} \left| \breve{f}^{(i)}_{k,z,t}(x) \right| < \infty,$$

since $c \geq 2$. Thus, applying (2.7) gives

(C-4)
$$\mathbb{E}\breve{f}_{k,z,t}(X_t^\varepsilon) = \sum_{i=0}^{n-k} \frac{t^i}{i!} L_\varepsilon^i \breve{f}_{k,z,t}(0) + \frac{t^{n-k+1}}{(n-k)!} \int_0^1 (1-\alpha)^{n-k} \mathbb{E}\{L_\varepsilon^{n-k+1} \breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)\} d\alpha.$$

To show that the last integral in (C-4) is bounded, we apply Lemma 2.2 to get that

$$\mathbb{E}\left\{(L_\varepsilon^{n-k+1}\breve{f}_{k,z,t})(X_{\alpha t}^\varepsilon)\right\} = \sum_{\mathbf{k}\in\mathcal{K}_{n-k+1}} \prod_{i=0}^{4} b_i^{k_i} \binom{n-k+1}{\mathbf{k}} \mathbb{E}\left[B_{\mathbf{k},\varepsilon}\breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)\right].$$

Thus, it is sufficient to show the boundedness of $\mathbb{E}B_{\mathbf{k},\varepsilon}\breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)$, for any $1 \leq k \leq n$ and $\mathbf{k} = (k_0, \ldots, k_4) \in \mathcal{K}_{n-k+1}$. Indeed, noting that (2.5) implies that

$$\tilde{M} := \int_{[0,1]^{k_3}\times\mathbb{R}^{k_3+k_4}} e^{\sum_{j=1}^{k_3}\beta_j w_j + \sum_{i=1}^{k_4} u_i} d\pi_{\mathbf{k},\varepsilon} < \infty,$$

we have, for any $x \in \mathbb{R}$ and some contants $K_1, K_2 < \infty$,

$$\begin{aligned}
\left|B_{\mathbf{k},\varepsilon}\breve{f}_{k,z,t}(x)\right| &\leq \int_{[0,1]^{k_3}\times\mathbb{R}^{k_3+k_4}} \left|\breve{f}_{k,z,t}^{(\ell_\mathbf{k})}\right| \left(x + \sum_{j=1}^{k_3}\beta_j w_j + \sum_{i=1}^{k_4} u_i\right) d\pi_{\mathbf{k},\varepsilon} \\
&\leq \int_{[0,1]^{k_3}\times\mathbb{R}^{k_3+k_4}} \mathbb{E}\left|\widetilde{f}_{k,z}^{(\ell_\mathbf{k})}\right| \left(U_t + x + \sum_{j=1}^{k_3}\beta_j w_j + \sum_{i=1}^{k_4} u_i\right) d\pi_{\mathbf{k},\varepsilon} \\
&\leq \tilde{M}\lambda_\varepsilon^{-k}\mathbb{E}\, e^{U_t} \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}} e^{z+x+\sum_{\ell=1}^{k} u_\ell} \bar{s}_\varepsilon(u_1)du_1 \prod_{\ell=2}^{k} \bar{s}_\varepsilon(u_\ell)du_\ell \\
&\quad + M_{i,\varepsilon}\lambda_\varepsilon^{-k} \sum_{j=0}^{\ell_\mathbf{k}-2} \int_{\mathbb{R}^{k-1}} \prod_{\ell=2}^{k} \bar{s}_\varepsilon(u_\ell)du_\ell \cdot \max_{0\leq j\leq i} \gamma_{j,\varepsilon/2} \\
&= M_1 e^x + M_2 < \infty,
\end{aligned}$$

where the third inequality follows from (C-3). It follows that $\mathbb{E}B_{\mathbf{k},\varepsilon}\breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)$ is $O_{\varepsilon,z_0}(1)$, and so is $\mathbb{E}L_\varepsilon^{n-k+1}\breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)$. Therefore, the last integral in (C-4) is indeed $O_{\varepsilon,z_0}(t^{n-k+1})$.

(4) By plugging (C-4) into (3.1) and (3.2) and rearranging terms, we obtain that

$$\begin{aligned}
\mathbb{E}f_z(Z_t) &= e^{-\lambda_\varepsilon t} \sum_{k=1}^{n} \frac{(\lambda_\varepsilon t)^k}{k!} \breve{f}_{k,z,t}(X_t^\varepsilon) + O_{\varepsilon,z_0}(t^{n+1}) \\
&= e^{-\lambda_\varepsilon t} \sum_{j=1}^{n} \frac{t^j}{j!} \sum_{k=1}^{j} \binom{j}{k} \lambda_\varepsilon^k L_\varepsilon^{j-k} \breve{f}_{k,z,t}(0) + O_{\varepsilon,z_0}(t^{n+1}). \qquad \text{(C-5)}
\end{aligned}$$

It remains to expand the coefficients

$$L_\varepsilon^{j-k}\breve{f}_{k,z,t}(0) = L_\varepsilon^{j-k}\left[\mathbb{E}\widetilde{f}_{k,z}(U_t + \cdot)\right](0) = \lambda_\varepsilon^{-k} L_\varepsilon^{j-k}\left[\mathbb{E}\widehat{f}_{k,z}(U_t + \cdot)\right](0). \qquad \text{(C-6)}$$

Using the expansion (3.6) and Remark 3.5, we have

$$\mathbb{E}\widehat{f}_{k,z}(U_t+x) = \sum_{i=0}^{n-j}\frac{t^i}{i!}\mathcal{L}^i\widehat{f}_{k,z}(x)+\frac{t^{n-j+1}}{(n-j+1)!}\int_0^1(1-\alpha)^{n-j}\mathbb{E}\Big(\mathcal{L}^{n-j+1}\widehat{f}_{k,z}(U_{\alpha t}+x)\Big)d\alpha$$

$$= \sum_{i=0}^{n-j}\frac{t^i}{i!}\sum_{l=0}^{i}B_l^i(y_0)\mathcal{L}_u^l\widehat{f}_{k,z}(x) + \frac{t^{n-j+1}}{(n-j+1)!}\int_0^1(1-\alpha)^{n-j}\mathbb{E}\left(\mathcal{L}^{n-j+1}\widehat{f}_{k,z}(U_{\alpha t}+x)\right)d\alpha. \tag{C-7}$$

Finally, by combining (C-5), (C-6) and (C-7), it follows that

$$\mathbb{E}f_z(Z_t) = e^{-\lambda_\varepsilon t}\sum_{j=1}^{n}\frac{t^j}{j!}\sum_{k=1}^{j}\binom{j}{k}\left[\sum_{i=0}^{n-j}\frac{t^i}{i!}L_\varepsilon^{j-k}\left(\sum_{l=0}^{i}B_l^i(y_0)\mathcal{L}_u^l\widehat{f}_{k,z}\right)(0)\right.$$

$$\left.+\frac{t^{n-j+1}}{(n-j+1)!}\int_0^1(1-\alpha)^{n-j}\mathbb{E}\Big\{L_\varepsilon^{j-k}\big[\mathcal{L}^{n-j+1}\widehat{f}_{k,z}(U_{\alpha t}+\cdot)\big](0)\Big\}d\alpha\right]+O_{\varepsilon,z_0}(t^{n+1}) \tag{C-8}$$

$$= e^{-\lambda_\varepsilon t}\sum_{j=1}^{n}\frac{t^j}{j!}\sum_{k=1}^{j}\binom{j}{k}\sum_{i=0}^{n-j}\frac{t^i}{i!}L_\varepsilon^{j-k}\left(\sum_{l=0}^{i}B_l^i(y_0)\mathcal{L}_u^l\widehat{f}_{k,z}\right)(0)+O_{\varepsilon,z_0}(t^{n+1}) \tag{C-9}$$

$$= e^{-\lambda_\varepsilon t}\sum_{j=1}^{n}\frac{t^j}{j!}\sum_{p+q+r=j}\binom{j}{p,q,r}L_\varepsilon^q\left(\sum_{m=0}^{r}B_m^r(y_0)\mathcal{L}_u^m\widehat{f}_{p,z}\right)(0)+O_{\varepsilon,z_0}(t^{n+1}). \tag{C-10}$$

Here in (C-9) we used the fact that the integral in (C-8) is $O_{\varepsilon,z_0}(1)$ as seen from the uniform boundedness condition (3.13) and the estimate (C-3). □

Appendix D. Proof of Theorem 6.1

We only consider $x>0$ (the case $x<0$ can be similarly analyzed by considering $\mathbb{P}(X_t\le x)$). Again, we start with the expression

$$\mathbb{P}(X_t\ge x) = \underbrace{e^{-\lambda_\varepsilon t}\mathbb{P}\left(X_t^\varepsilon\ge x\right)}_{B_t(x)}+\underbrace{e^{-\lambda_\varepsilon t}\sum_{k=n+1}^{\infty}\frac{(\lambda_\varepsilon t)^k}{k!}\mathbb{P}\left(X_t^\varepsilon+\sum_{i=1}^{k}\xi_i\ge x\right)}_{C_t(x)} \tag{D-1}$$

$$+\underbrace{e^{-\lambda_\varepsilon t}\sum_{k=1}^{n}\frac{(\lambda_\varepsilon t)^k}{k!}\mathbb{P}\left(X_t^\varepsilon+\sum_{i=1}^{k}\xi_i\ge x\right)}_{D_t(x)}. \tag{D-2}$$

Let us denote by f_t^ε the density of X_t^ε, whose existence follows from (6.4). Given that

$$\frac{d}{dx}\mathbb{P}\left(X_t^\varepsilon+\sum_{i=1}^{k}\xi_i\ge x\right)=-\frac{1}{\lambda_\varepsilon^k}f_t^\varepsilon*\bar{s}_\varepsilon^{*k}(x),$$

$$\text{and that }\sup_x|f_t^\varepsilon*\bar{s}_\varepsilon^{*k}(x)|\le\sup_x|\bar{s}_\varepsilon^{*k}(x)|\le\gamma_{\varepsilon/2,0}\lambda_\varepsilon^{k-1},$$

one can interchange derivative and summation in (D-1) to show that $C_t(x)$ admits a density $c_t(x)$ and moreover,

(D-3)
$$\sup_x |c_t(x)| = \sup_x e^{-\lambda_\varepsilon t} \sum_{k=n+1}^{\infty} \frac{t^k}{k!} f_t^\varepsilon * \bar{s}_\varepsilon^{*k}(x) \leq e^{-\lambda_\varepsilon t} \frac{\gamma_{\varepsilon/2,0}}{\lambda_\varepsilon} \sum_{k=n+1}^{\infty} \frac{(\lambda_\varepsilon t)^k}{k!} \leq \lambda_\varepsilon^n \gamma_{\varepsilon/2,0} t^{n+1}.$$

Also, in view of Proposition III.2 in [27], there exists a real $\varepsilon_0(\eta, n) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $t \leq 1$,

$$\sup_{|x| \geq \eta} f_t^\varepsilon(x) \leq M(\eta, \varepsilon) t^{n+1}, \tag{D-4}$$

where $M(\eta, \varepsilon)$ is some unversal constant depending only on η and ε. The last step is to deal with the terms in D_t. Recall that

$$\mathbb{P}\left(X_t^\varepsilon + \sum_{i=1}^k \xi_i \geq x\right) = \mathbb{E}\widetilde{f}_{k,x}\left(X_t^\varepsilon\right),$$
$$\frac{d^{(i)}}{dz^i} \widetilde{f}_{k,x}(y) = \lambda_\varepsilon^{-k}(-1)^{i-1} \bar{s}_\varepsilon^{*(k-1)} * \bar{s}_\varepsilon^{(i-1)}(x-y),$$

with

$$\widetilde{f}_{k,x}(y) := \mathbb{P}\left(y + \sum_{\ell=1}^k \xi_i \geq x\right) = \lambda_\varepsilon^{-k} \int_{x-y}^{\infty} \bar{s}_\varepsilon^{*k}(u) du.$$

Then, applying the iterated formula (2.7), we get

(D-5)
$$\mathbb{E}\widetilde{f}_{k,x}(X_t^\varepsilon) = \sum_{i=0}^{n-k} \frac{t^i}{i!} L_\varepsilon^i \widetilde{f}_{k,x}(0) + \frac{t^{n+1-k}}{(n-k)!} \int_0^1 (1-\alpha)^{n-k} \mathbb{E}\left(L_\varepsilon^{n+1-k} \widetilde{f}_{k,x}(X_{\alpha t}^\varepsilon)\right) d\alpha.$$

Using the representation of L_ε in Lemma 2.2, one can easily verify that

$$\frac{d}{dx} L_\varepsilon^i \widetilde{f}_{k,x}(y) = -L_\varepsilon^i \widetilde{f}'_{k,x}(y) = -(\lambda_\varepsilon)^{-k} L_\varepsilon^i \hat{s}_{k,x}(y), \tag{D-6}$$

$$\sup_{x,z} \left| \frac{d}{dx} L_\varepsilon^{n+1-k} \widetilde{f}_{k,x}(y) \right| \leq M_{n,k,\varepsilon} \max_{0 \leq k \leq 2n} \{\gamma_{\varepsilon/2,k}\}, \tag{D-7}$$

for some constants $M_{n,k,\varepsilon} < \infty$. Hence, one can pass d/dx through the integral and the expectation in the last term of (D-5) to get

$$\frac{d}{dx} \mathbb{E}\widetilde{f}_{k,x}(X_t^\varepsilon) = -(\lambda_\varepsilon)^{-k} \sum_{i=0}^{n-k} \frac{t^i}{i!} L_\varepsilon^i \hat{s}_{k,x}(0) + t^{n+1-k} O_{\varepsilon,k,n}(1), \tag{D-8}$$

where $O_{\varepsilon,k,n}(1)$ means that $\sup_x |O_{\varepsilon,k,n}(1)|$ is bounded by a constant depending only on ε, k, and n. Differentiating $\mathbb{P}(X_t \geq x)$ in (D-1) and plugging (D-3), (D-4), (D-8), we get that for any $0 < \varepsilon < \varepsilon_0$ and $t \leq 1$,

$$f_t(x) = e^{-\lambda_\varepsilon t} \sum_{k=1}^n \frac{t^k}{k!} \sum_{i=0}^{n-k} \frac{t^i}{i!} L_\varepsilon^i \hat{s}_{k,x}(0) + t^{n+1} O_{\varepsilon,\eta}(1),$$

where $O_{\varepsilon,\eta}(1)$ is such that $\sup_{t \leq 1} \sup_{|x| \geq \eta} |O_{\varepsilon,\eta}(1)| < \infty$. Rearranging the terms above, we have

$$f_t(x) = e^{-\lambda_\varepsilon t} \sum_{p=1}^n c_p(x) \frac{t^p}{p!} + t^{n+1} O_{\varepsilon,\eta}(1),$$

with

$$c_p(x) := \sum_{k=1}^{p} \binom{p}{k} L_\varepsilon^{p-k} \hat{s}_{k,x}(0).$$

The expression in (6.10) follows from the Taylor expansion of $e^{-\lambda_\varepsilon t}$, using also that $\sup_x |c_p(x)| < \infty$ (a fact that itself follows from (D-6)). Finally, the "constant property" of (6.10), for any $0 < \varepsilon < \varepsilon_0$, follows from inversion. Indeed, given that a posterior

$$f_t(x) = \sum_{k=1}^{n} \frac{a_k(x)}{k!} t^k + t^{n+1} O_{\eta,\varepsilon}(1) \tag{D-9}$$

holds true for any $t \leq 1$ and $0 < \varepsilon < \varepsilon_0$, $a_k(x)$ can be recovered from $f_t(x)$ (independently of ε) by the recursive formulas:

$$a_1(x) = \lim_{t\to 0} \frac{1}{t} f_t(x), \quad a_k(x) = \lim_{t \to 0} \frac{k!}{t^k} \left(f_t(x) - \sum_{i=1}^{k} \frac{a_i(x)}{i!} t^i \right), \quad 2 \leq k \leq n.$$

□

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