SMALL-TIME EXPANSIONS OF THE DISTRIBUTIONS, DENSITIES, AND OPTION PRICES OF STOCHASTIC VOLATILITY MODELS WITH LÉVY JUMPS

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Abstract. We consider a stochastic volatility model with Lévy jumps for a log-return process $Z = (Z_t)_{t\geq 0}$ of the form $Z = U + X$, where $U = (U_t)_{t\geq 0}$ is a classical stochastic volatility process and $X = (X_t)_{t\geq 0}$ is an independent Lévy process with absolutely continuous Lévy measure ν. Small-time expansions, of arbitrary polynomial order, in time-t, are obtained for the tails $\mathbb{P}(Z_t \geq z)$, $z > 0$, and for the call-option prices $\mathbb{E}\left(e^{z+Z_t} - 1\right)_+$, $z \neq 0$, assuming smoothness conditions on the density of ν away from the origin and a small-time large deviation principle on U. Our approach allows for a unified treatment of general payoff functions of the form $\varphi(x)\mathbf{1}_{x\geq z}$ for smooth functions φ and $z > 0$. As a consequence of our tail expansions, the polynomial expansions in t of the transition densities f_t are also obtained under mild conditions.

1. Introduction

It is generally recognized that accurate modeling of the option market and asset prices requires a mixture of a continuous diffusive component and a jump component. For instance, based on high-frequency statistical methods for Itô semimartingales, several empirical studies have statistically rejected the null hypothesis of either a purely-jump or a purely-continuous model (see, e.g., [1], [4], [5], [6], [32]). Similarly, based on partially heuristic arguments, [10] characterizes the small-time behavior of at-the-money (ATM) and out-of-the-money (OTM) call option prices, and based on these results, then argues that both, a continuous and a jump component, are needed to explain the behavior of the market implied volatilities for S&P500 index options. Another empirical work in the same vein is [30], where a small-time small-log-moneyness approximation for the implied volatility surface was studied in the case of a local jump-diffusion model with finite jump activity. Based on S&P500 option market data, [30] also concludes that jumps are significant in the risk-neutral price dynamics of the underlying asset.

Historically, local volatility models (and more recently stochastic volatility models) were the models of choice to replicate the skewness of the market implied volatilities at a given time (see [19] and [21] for more details). However, it is a well-known empirical fact that implied volatility skewness is more accentuated as the expiration time approaches. Such a phenomenon is hard to reproduce within the purely-continuous framework unless the "volatility of volatility" is forced to

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take very high values. Furthermore, as nicely explained in [11] (Chapter 1), the very existence of a market for short-term options is evidence that option market participants operate under the assumption that a jump component is present.

In recent years the literature of small-time asymptotics for vanilla option prices of jump-diffusion models has grown significantly with strong emphasis to consider either a purely-continuous model or a purely Lévy model. In case of stochastic volatility models and local volatility models, we can mention, among others, [7], [8], [12], [17], [18], [20], [23], [33]. For Lévy processes, [36] and [39] show that OTM option prices are generally[1] asymptotically equivalent to the time-to-maturity τ as $\tau \to 0$. In turn, such a behavior implies that the implied volatilities of a Lévy model explodes as $\tau \to 0$. The exact first order asymptotic behavior of the implied volatility close to maturity was independently obtained in [14] and [39], while the former paper also gives the second order asymptotic behavior. There are very few articles that consider simultaneously stochastic volatility and jumps in the model. One such work is [10] which obtains, partially via heuristic arguments, the first order asymptotic behavior of an Itô semimartingale with jumps. Concretely, ATM option prices of pure-jump models of bounded variation decrease at the rate $O(\tau)$, while they are just $O(\sqrt{\tau})$ under the presence of a Brownian component. By considering a stable pure-jump component, [10] also shows that, in general, the rate could be $O(\tau^{\beta})$, for some $\beta \in (0,1)$. For OTM options, it is also argued that the first order behavior is $O(e^{-c/\tau})$ in the purely-continuous case, while it becomes $O(\tau)$ under the presence of jumps. Very recently, [31] formally shows that the leading term of ATM option prices is of order $\sqrt{T}$ for a relatively general class of purely-continuous Itô models, while for a more general type of Itô processes with α-stable-like small jumps, the leading term is $O(\tau^{1/\alpha})$ (see also [14, Proposition 4.2] and [39, Proposition 5] for related results in exponential Lévy models). Fractional expansions are also obtained for the distribution functions of some Lévy processes in [29].

In this article, we consider a jump diffusion model by combining a stochastic volatility model with a pure-jump Lévy process of infinite jump activity. More precisely, we consider a market consisting of a risk-free asset with constant interest rate $r \geq 0$ and a risky asset with price process $(S_t)_{t\geq 0}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We assume that, under $\mathbb{Q}$, the log-return process

$$Z_t := \log \frac{e^{-rt} S_t}{S_0}, \tag{1.1}$$

follows the jump diffusion model

$$Z_t = U_t + X_t, \quad dU_t = \mu(Y_t)dt + \sigma(Y_t)dW_t^{(1)}, \tag{1.2}$$

$$dY_t = \alpha(Y_t)dt + \gamma(Y_t)dW_t^{(2)} \tag{1.3}$$

with $U_0 = X_0 = 0$, $Y_0 = y_0 \in \mathbb{R}$. Here, $W^{(1)}$ and $W^{(2)}$ are Wiener processes adapted to $(\mathcal{F}_t)$, X is an independent $(\mathcal{F}_t)$-adapted pure-jump Lévy process with triplet $(b, 0, \nu)$, and σ, γ, μ and α are deterministic functions such that (1.2)-(1.3)

[1] That is, except for some pathological cases (see [36] for examples)

admits a solution. We also assume that $\int_{z>1} e^z \nu(dz) < \infty$ and

$$b = -\int_{\mathbb{R}} \left(e^z - 1 - z\mathbf{1}_{|z|\leq 1}\right) \nu(dz), \quad \text{and} \quad \mu(y) = -\frac{1}{2}\sigma^2(y). \tag{1.4}$$

In particular, note that $\mathbb{Q}$ is assumed to be a martingale measure (i.e. $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale). The model (1.2)-(1.3) is appealing in practice since it incorporates jumps in the asset price process as well as volatility clustering and leverage effects. The process $(Y_t)_{t\geq 0}$ is the underlying volatility factor driving the stochastic volatility of the process.

For $z \neq 0$ and $t > 0$, let

$$G_t(z) := \mathbb{E}\left(e^{z+Z_t} - 1\right)_+, \tag{1.5}$$

where $\mathbb{E}$ denotes, from now on, the expectation under $\mathbb{Q}$. We will show that, under mild conditions, the following small-time expansions for G_t hold true:

$$G_t(z) = \sum_{j=0}^{n} b_j(z)\,\frac{t^j}{j!} + O(t^{n+1}), \tag{1.6}$$

for each $n \geq 0$ and certain functions b_j. Note that the time-t price of a European call option with strike K, which is not at-the-money, when the spot stock price and the underlying volatility factor have respective values s and y_0, can then be expressed as

$$C(t,s) := e^{-r(T-t)}\mathbb{E}\left(\left(S_T - K\right)_+ \middle| \mathcal{F}_t\right) = Ke^{-r\tau}G_\tau(\ln s - \ln K), \tag{1.7}$$

where $\tau = T - t$ and $s \neq K$. Hence, the small-time behavior of (1.5) leads to close-to-expiry approximations for the price of an arbitrary not-at-the-money call option as a polynomial expansion in time-to-maturity. From (1.1), note also that the expression

$$S_0 e^{-z} G_t(z) = e^{-rt}\mathbb{E}\left(S_t - S_0 e^{rt-z}\right)_+ \tag{1.8}$$

can be interpreted as the time-t call option price with log-moneyness[2] $\kappa := -z$.

Small-time option price asymptotics for the model (1.2)-(1.3) were also considered in [30], but only for finite-jump activity Lévy processes X. Another important difference is that, here, we focus on small-time asymptotics for fixed log-moneyness $z \neq 0$, while [30] considers approximations where z is simultaneously made to converge to 0 as $t \to 0$ (small-time and small-log-moneyness asymptotics). Let us also remark that [30] assumes throughout (and without proof) that the implied volatility surface satisfies an expansion in powers of z and t, which, in our opinion, is a rather strong assumption. Another related work is [31], where the first order small-time small-log-moneyness asymptotics is considered for a class of Itô semimartingales with non-zero continuous component (Theorem 3.1 therein). For the CGMY and related models, higher order approximations for ATM option prices are obtained in [15].

Our method of proof is built on a type of iterated Dynkin formula of the form

$$\mathbb{E}g(X_t) = g(0) + \sum_{k=1}^{n} \frac{t^k}{k!} L^k g(0) + \frac{t^{n+1}}{n!} \int_0^1 (1-\alpha)^n \mathbb{E}\left(L^{n+1} g(X_{\alpha t})\right) d\alpha, \tag{1.9}$$

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[2]As usual, log-moneyness is defined as the logarithm of the ratio between the strike K and the forward price $S_0 e^{rt}$.

where g is a sufficiently smooth function and L is the infinitesimal generator of the Lévy process X given by

$$Lg(x) := \frac{\sigma^2}{2} g''(x) + bg'(x) + \int \left(g(z+x) - g(x) - zg'(x)\mathbf{1}_{\{|z|\leq 1\}} \right) \nu(dz), \tag{1.10}$$

for $g \in C_b^2$ and a Lévy triplet (b, σ^2, ν) (see Section 2.1 below for terminology). The main complication with option call prices arises from the lack of smoothness of the payoff function $g_z(x) = (e^{z+x} - 1)_+$. In order to "regularize" the payoff function g, we follow a two step procedure. First, we decompose the Lévy process into a compound Poisson process with a smooth jump density vanishing in a neighborhood of the origin and an independent Lévy process with small jumps. Then, we condition $\mathbb{E}g(X_t)$ on the number of jumps of the compound Poisson component of X and apply the Dynkin's formula on each of the resulting terms. Contrary to the approaches in [14] and [39], where the special form of the payoff function $g_z(x) = (e^{z+x} - 1)_+$ plays a key role, our approach can handle more general payoff functions of the form

$$g_z(x) = \varphi(x)\mathbf{1}_{\{x \geq z\}}, \tag{1.11}$$

for a smooth function φ and positive z. By considering the particular case $\varphi(x) \equiv 1$, we generalize the distribution expansions in [13] to our jump-diffusion setting. Let us emphasize that the process Z in (1.2) is not truly a Markov model but rather a hidden Markov model. This fact causes some technical subtleties that require a careful analysis of the iterated infinitesimal generator of the bivariate Markov process $\{(U_t, Y_t)\}_{t \geq 0}$.

As an equally relevant second contribution of our paper, we also obtain polynomial expansions for the transition densities f_t of the Lévy process, under conditions involving only the Lévy density of X. This is an important improvement to our former results in [13], where a uniform boundedness condition on all the derivatives of f_t away from the origin was imposed. Expansions for the transition densities of local volatility models (with possibly jumps but only of finite activity) have appeared before in the literature (e.g. see [2], [3], [40]). Unlike our approach, the general idea in these papers consists of first proposing the general form of the expansion, and then choosing the coefficients so that either the backward or forward Kolmogorov equation is satisfied. The resulting coefficients typically involve iterated infinitesimal generators as in our expansions, even though our approximations are uniform away from the origin.

The paper is organized as follows. Section 2 contains some preliminary results on Lévy processes, which will be needed throughout the paper. Section 3 establishes the small-time expansions, of arbitrary polynomial order in t, for both the tail distributions $\mathbb{P}(Z_t \geq z)$, $z > 0$, and the call-option price function $G_t(z)$, $z \neq 0$. This section also justifies the validity of our results for payoff functions of the form (1.11). Section 4 illustrates our approach by presenting the first few terms of those expansions. Interestingly enough, the first two coefficients of the expansion of the general model coincide with the first two coefficients of an exponential Lévy model. Section 5 obtains the asymptotic behavior of the corresponding implied volatility. Section 6 gives a small-time expansion for the transition density of a general Lévy process under rather mild conditions. The proofs of our main results are deferred to Appendices.

2. Background and preliminary results

2.1. Notation.

Throughout this paper, C^n (or $C^n(\mathbb{R})$), $n \geq 0$, is the class of real valued functions, defined on $\mathbb{R}$, which have continuous derivatives of order $0 \leq k \leq n$, while $C_b^n \subset C^n$ corresponds to those functions having bounded derivatives. In a similar fashion, C^∞ (or $C^\infty(\mathbb{R})$) is the class of real valued functions, defined on $\mathbb{R}$, which have continuous derivatives of any order $k \geq 0$, while $C_b^\infty(\mathbb{R}) \subset C^\infty$ are again the ones having bounded derivatives. Sometimes $\mathbb{R}$ will be replaced by $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ or $\mathbb{R}^k$ when the functions are defined on these spaces.

Throughout this section, X denotes a Lévy process with triplet (b, σ^2, ν) defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$. Let us write X in terms of its Lévy-Itô decomposition:

$$X_t = bt + \sigma W_t + \int_0^t \int_{|z|>1} z\mu(ds, dz) + \int_0^t \int_{|z|\leq 1} z\bar{\mu}(ds, dz),$$

where W is a Wiener process and μ is an independent Poisson measure on $\mathbb{R}_+ \times \mathbb{R}\backslash\{0\}$ with mean measure $dt\nu(dz)$ and compensator $\bar{\mu}$. For each $\varepsilon > 0$, let $c_\varepsilon \in C^\infty$ be a symmetric truncation function such that $\mathbf{1}_{[-\varepsilon/2,\varepsilon/2]}(z) \leq c_\varepsilon(z) \leq \mathbf{1}_{[-\varepsilon,\varepsilon]}(z)$ and let $\bar{c}_\varepsilon := 1 - c_\varepsilon$. Next, for $0 < \varepsilon < 1$, consider two independent Lévy processes, denoted by $\bar{X}^\varepsilon$ and X^ε, with respective Lévy triplets $(0, 0, \bar{c}_\varepsilon(z)\nu(dz))$ and $(b_\varepsilon, \sigma^2, c_\varepsilon(z)\nu(dz))$, where

$$b_\varepsilon := b - \int_{|z|\leq 1} z\bar{c}_\varepsilon(z)\nu(dz).$$

Note that $(X_t)_{t\geq 0}$ has the same law as $(X_t^\varepsilon + \bar{X}_t^\varepsilon)_{t\geq 0}$ and, therefore, without loss of generality, we assume hereafter that $X = X^\varepsilon + \bar{X}^\varepsilon$. Note also that $\bar{X}^\varepsilon$ is a compound Poisson process with intensity of jumps $\lambda_\varepsilon := \int \bar{c}_\varepsilon(z)\nu(dz)$ and jumps distribution $\bar{c}_\varepsilon(z)\nu(dz)/\lambda_\varepsilon$. Throughout, $(\xi_i^\varepsilon)_{i\geq 1}$ stands for the jumps of the process $\bar{X}^\varepsilon$. The process X^ε has infinitesimal generator L_ε given by

$$L_\varepsilon g(y) = b_\varepsilon g'(y) + \frac{\sigma^2}{2} g''(y) + \mathcal{I}_\varepsilon g(y), \tag{2.1}$$

for $g \in C_b^2$, where

$$\begin{aligned}\mathcal{I}_\varepsilon g(y) &:= \int_{\mathbb{R}_0} \left\{ g(y+z) - g(y) - zg'(y)\mathbf{1}_{|z|\leq 1} \right\} c_\varepsilon(z)\nu(dz) \\ &= \int_{\mathbb{R}_0} \int_0^1 g''(y + \beta w)(1-\beta)d\beta w^2 c_\varepsilon(w)\nu(dw).\end{aligned}$$

The following tail estimate for X^ε is also used in the sequel:

$$\mathbb{P}(|X_t^\varepsilon| \geq z) \leq t^{az} \exp(az_0 \ln z_0) \exp(az - az \ln z), \tag{2.2}$$

where $a \in (0, \varepsilon^{-1})$, and $t, z > 0$ satisfy $t < z/z_0$ for some z_0 depending only on a (see [38, Section 2.6], [37, Lemma 3.2] and [13, Remark 3.1] for proofs and extensions).

Throughout the paper, we also make the following standing assumptions:

$$\nu(dz) = s(z)dz, \; s \in C^\infty(\mathbb{R}\backslash\{0\}) \;\text{ and }\; \gamma_{k,\delta} := \sup_{|z|>\delta} |s^{(k)}(z)| < \infty, \;\text{ for all } \delta > 0, \tag{2.3}$$

$$\int_{|z|>1} e^{c|z|}\nu(dz) < \infty, \;\text{ for some } c > 2. \tag{2.4}$$

Finally, the following notation are also in use:

$$s_\varepsilon := c_\varepsilon s, \quad \bar{s}_\varepsilon := (1-c_\varepsilon)s, \quad L^0 g = g, \quad L^{k+1}g = L(L^k g), \quad (k \geq 0),$$
$$\bar{s}_\varepsilon^{*0} * g = g, \quad \bar{s}_\varepsilon^{*1} = \bar{s}_\varepsilon, \quad \bar{s}_\varepsilon^{*k}(x) = \int \bar{s}_\varepsilon^{*(k-1)}(x-u)\bar{s}_\varepsilon(u)du, \quad (k \geq 2).$$

2.2. **Dynkin's formula for smooth subexponential functions.**

Let us recall that taking expectations in the well-known Dynkin formula gives:

$$\mathbb{E}g(X_t) = g(0) + \int_0^t \mathbb{E}\left(Lg(X_u)\right)du = g(0) + t\int_0^1 \mathbb{E}\left(Lg(X_{\alpha t})\right)d\alpha, \tag{2.5}$$

valid if $g \in C_b^2$. Iterating (2.5), one obtains the following expansion for $g \in C_b^{2n+2}$ (e.g., see [24, Proposition 9]):

$$\mathbb{E}g(X_t) = g(0) + \sum_{k=1}^n \frac{t^k}{k!}L^k g(0) + \frac{t^{n+1}}{n!}\int_0^1 (1-\alpha)^n \mathbb{E}\left(L^{n+1}g(X_{\alpha t})\right)d\alpha. \tag{2.6}$$

Expansions of the form (2.6) are then called iterated-type Dynkin expansions. For our purposes, it will be useful to extend (2.6) to subexponential functions. The proof of the following result can be found in Appendix A.

Lemma 2.1. *Let ν satisfy (2.4), and let $g \in C^{2n+2}$ be such that*

$$\limsup_{|y|\to\infty} e^{-\frac{c}{2}|y|}|g^{(i)}(y)| < \infty, \tag{2.7}$$

for any $0 \leq i \leq 2n+2$. Then, (2.6) holds true.

Below, let

$$b_0 := -\int_{\mathbb{R}} \bar{c}_\varepsilon(u)\nu(du), \quad b_1 := b - \int_{\mathbb{R}} u(c_\varepsilon(u) - \mathbf{1}_{|u|\leq 1})\nu(du),$$
$$b_2 := \sigma^2/2, \quad b_3 := \frac{1}{2}\int_{\mathbb{R}} u^2 c_\varepsilon(u)\nu(du), \quad \text{and} \quad b_4 := \int_{\mathbb{R}} \bar{c}_\varepsilon(u)\nu(du).$$

Note that all these constants depend on $\varepsilon > 0$, but this is not explicitly indicated for the ease of notation. In order to work with the iterated infinitesimal generator L^k appearing in (2.6), the forthcoming representation will turn out to be useful (see [13, Lemma 4.1] for its verification[3]).

Lemma 2.2. *Let $\mathcal{K}_k = \{\mathbf{k} = (k_0, \ldots, k_4) \in \mathbb{N}^5 : k_0 + \cdots + k_4 = k\}$ and for $\mathbf{k} \in \mathcal{K}_k$, let $\ell_{\mathbf{k}} := k_1 + 2k_2 + 2k_3$. Then, for any $k \geq 1$ and $\varepsilon > 0$,*

$$L^k g(x) = \sum_{\mathbf{k}\in\mathcal{K}_k} \prod_{i=0}^4 b_i^{k_i} \binom{k}{\mathbf{k}} B_{\mathbf{k},\varepsilon} g(x), \tag{2.8}$$

where

$$B_{\mathbf{k},\varepsilon} g(x) := \begin{cases} \int g^{(\ell_{\mathbf{k}})}\left(x + \sum_{j=1}^{k_3}\beta_j w_j + \sum_{i=1}^{k_4} u_i\right) d\pi_{\mathbf{k},\varepsilon}, & \text{if } k_3 + k_4 > 0, \\ g^{(\ell_{\mathbf{k}})}(x), & \text{if } k_3 = k_4 = 0, \end{cases}$$

[3] For convenience we switched the role of b_3 and b_4.

and where the above integral is with respect to the probability measure

$$d\pi_{\mathbf{k},\varepsilon} = \prod_{j=1}^{k_3} \frac{1}{b_3} c_\varepsilon(w_j) w_j^2 \nu(dw_j)(1-\beta_j)d\beta_j \prod_{i=1}^{k_4} \frac{1}{b_4} \bar{c}_\varepsilon(u_i)\nu(du_i),$$

on $\mathbb{R}^{k_3} \times [0,1]^{k_3} \times \mathbb{R}^{k_4}$ *(under the standard conventions that* $0/0 = 1$ *and* $\prod_{i=1}^{0} = 1$*).*

Remark 2.3. The expansion (2.8) holds true for (possibly unbounded) functions $g \in C^{2k+2}$ satisfying (2.7) for any $0 \le i \le 2k+2$.

3. Small-time expansions for the tail distributions and option prices

In this section, we derive the small-time expansions for both the tail distribution $\mathbb{P}(Z_t \ge z)$, $z > 0$, and for the call-option price function $\mathbb{E}\left(e^{z+Z_t} - 1\right)_+$, $z \neq 0$. With an approach similar to that in [13, Theorem 3.2], the idea is to apply the following general moment expansion (easily obtained by conditioning on the number of jumps of the process $\bar{X}_t^\varepsilon$ introduced in Section 2.1):

$$\mathbb{E}f(Z_t) = e^{-\lambda_\varepsilon t}\mathbb{E}f\left(U_t + X_t^\varepsilon\right) + e^{-\lambda_\varepsilon t} \sum_{k=n+1}^{\infty} \frac{(\lambda_\varepsilon t)^k}{k!} \mathbb{E}f\left(U_t + X_t^\varepsilon + \sum_{i=1}^{k} \xi_i^\varepsilon\right) \tag{3.1}$$

$$+ e^{-\lambda_\varepsilon t} \sum_{k=1}^{n} \frac{(\lambda_\varepsilon t)^k}{k!} \mathbb{E}f\left(U_t + X_t^\varepsilon + \sum_{i=1}^{k} \xi_i^\varepsilon\right), \tag{3.2}$$

where $\{\xi_i^\varepsilon\}_{i\ge1}$ are the jumps of the process $\bar{X}^\varepsilon$. We shall take $f(u) = f_z(u) := \mathbf{1}_{\{u\ge z\}}$ in order to obtain the expansion of the transition distribution and $f(u) = f_z(u) := (e^{z+u} - 1)_+$ in order to obtain the expansion of the call-option price. To work out the terms in (3.2), we use the iterated formula (2.6), while to estimate the terms in (3.1), we assume that the underlying stochastic volatility model U satisfies a short-time "large deviation principle" of the form:

$$\lim_{t\to0} t \ln \mathbb{P}(U_t > u) = -\frac{1}{2}d(u)^2, \quad (u > 0), \tag{3.3}$$

where $d(u)$ is a strictly positive rate function. In Section 3.4 we review conditions for (3.3) to hold.

The expansions provided in the sequel will hold uniformly outside a neighborhood of the origin. Concretely, for fixed $z_0 > 0$ and $\varepsilon > 0$, the term $O_{\epsilon,z_0}(t^j)$ denotes a quantity, depending on z, ε, and t, such that

$$\sup_{0<t\le t_0} \sup_{|z|\ge z_0} t^{-j}|O_{\varepsilon,z_0}(t^j)| < \infty, \tag{3.4}$$

for some $t_0 > 0$, small enough, depending itself on ε and z_0.

3.1. **Expansions for the tail distributions.**

We first treat the case $f_z(u) := \mathbf{1}_{\{u\ge z\}}$. The following expansion for the tail distributions of Z (whose proof can be found in Appendix B) holds true.

Theorem 3.1. *Let $z_0 > 0$, $n \geq 1$, and $0 < \varepsilon < z_0/(n+1) \wedge 1$. Let the dynamics of Z be given by (1.2) and (1.3), and the conditions (2.3)-(2.4) and (3.3) be satisfied. Then, there exists $t_0 > 0$ such that, for any $z \geq z_0$ and $0 < t < t_0$,*

$$\mathbb{P}(Z_t \geq z) = e^{-\lambda_\varepsilon t} \sum_{j=1}^{n} \widehat{A}_{j,t}(z) \frac{t^j}{j!} + O_{\varepsilon, z_0}(t^{n+1}), \tag{3.5}$$

where

$$\widehat{A}_{j,t}(z) := \sum_{k=1}^{j} \binom{j}{k} \mathbb{E}\left(\left(L_\varepsilon^{j-k} \widehat{f}_{k,z} \right)(U_t) \right),$$

with $\widehat{f}_{k,z}(y) := \int_{z-y}^{\infty} \bar{s}_\varepsilon^{*k}(u) du$.

The expression (3.5) is not really satisfactory since the coefficients $\widehat{A}_{j,t}$ are time-dependent and so the asymptotic behavior of the tail probability $\mathbb{P}(Z_t \geq z)$ as $t \to 0$ is unclear. In order to obtain an expansion of $\widehat{A}_{j,t}$, we can further obtain an iterated Dynkin expansion for $\mathbb{E}g(U_t, Y_t)$ and a suitable moment function g. Indeed, assuming for simplicity that $W^{(1)}$ and $W^{(2)}$ are independent, (U, Y) is a Markov process with infinitesimal generator

$$\mathcal{L}g(u, y) = \mu(y) \frac{\partial g}{\partial u} + \frac{\sigma^2(y)}{2} \frac{\partial^2 g}{\partial u^2} + \alpha(y) \frac{\partial g}{\partial y} + \frac{\gamma^2(y)}{2} \frac{\partial^2 g}{\partial y^2}, \tag{3.6}$$

for $g \in C_b^2$. Itô's formula and induction imply that

$$\mathbb{E}g(U_t, Y_t) = g(u_0, y_0) + \sum_{k=1}^{n} \frac{t^k}{k!} \mathcal{L}^k g(u_0, y_0) + \frac{t^{n+1}}{n!} \int_0^1 (1-\alpha)^n \mathbb{E}\left\{ \mathcal{L}^{n+1} g(U_{\alpha t}, Y_{\alpha t}) \right\} d\alpha, \tag{3.7}$$

for any function g such that $\mathcal{L}^k g(u, y)$ is well-defined and belongs to C_b for any $0 \leq k \leq 2n+2$. As in the case of the infinitesimal generator of X, one can view the operator (3.6) as the sum of four operators. However, given that in general those operators do not commute, it is not possible to write a simple closed-form expression for $\mathcal{L}^k g(u_0, y_0)$ as is the case for X. Nevertheless, the following result gives a recursive method to get such an expression when $\mu(y) = -\sigma^2(y)/2$ and $g(u, y) = h(u)$ as needed here .

Proposition 3.2. *Let the dynamics of U and Y be given by (1.2) and (1.3) with independent $W^{(1)}$ and $W^{(2)}$ and with C^∞ deterministic functions α, σ^2, and γ^2. For $h, \tilde{h} \in C^2(\mathbb{R})$, let*

$$\mathcal{L}_1 h(u) := h''(u) - h'(u), \quad \mathcal{L}_2 \tilde{h}(y) := \frac{\gamma^2(y)}{2} \tilde{h}''(y) + \alpha(y) \tilde{h}'(y). \tag{3.8}$$

Then, for $h \in C^{2n+2}$ and $g(u, y) := h(u)$, the infinitesimal generator (3.6) is such that

$$\mathcal{L}^k g(u, y) = \sum_{j=0}^{k} B_j^k(y) \mathcal{L}_1^j h(u), \quad \text{for } 0 \leq k \leq n,$$

where $B_j^k(y)$ are defined iteratively as follows:

$$B_0^0(y) = 1, \quad B_j^k(y) = 0, \quad \forall j \notin \{1, \dots, k\}, \quad k \geq 1,$$

$$B_j^k(y) = \mathcal{L}_2 B_j^{k-1}(y) + \frac{\sigma^2(y)}{2} B_{j-1}^{k-1}(y), \quad 1 \leq j \leq k, \quad k \geq 1.$$

Proof. The proof is done by induction. □

Using the previous result, we can easily find conditions for the iterated formula (3.7) to hold. To this end, let us introduce the following class of functions:

$$\mathcal{C}_l^n = \big\{p \in C^n : |p^{(i)}(x)| \leq \mathcal{M}_n(1+|x|), \text{ for all } 0 \leq i \leq n, \text{ and} \\ \text{for some } \mathcal{M}_n < \infty \text{ independent of } x\big\}.$$

Corollary 3.3. *In addition to the conditions of Proposition 3.2, let $\gamma \in \mathcal{C}_l^0$ and let $\alpha, \sigma^2, \gamma^2 \in \mathcal{C}_l^k$, for any $k \geq 0$. Then, (3.7) is satisfied for $g(u,y) := h(u)$ whenever $h \in C_b^{2n+2}$.*

Proof. Using Itô's formula and induction, (3.7) will hold provided that

$$\int_0^t \frac{\partial \mathcal{L}^n g(U_s,Y_s)}{\partial u}\sigma(Y_s)dW_s^{(1)}, \quad \text{and} \quad \int_0^t \frac{\partial \mathcal{L}^n g(U_s,Y_s)}{\partial y}\gamma(Y_s)dW_s^{(2)},$$

are true martingales. For this, it suffices that

$$\mathbb{E}\int_0^t \left(\frac{\partial \mathcal{L}^n g}{\partial u}\sigma\right)^2 ds < \infty, \quad \text{and} \quad \mathbb{E}\int_0^t \left(\frac{\partial \mathcal{L}^n g}{\partial y}\gamma\right)^2 ds < \infty, \quad \forall t \geq 0.$$

Let us recall that since α and γ belong to $\mathcal{C}_l^0$,

$$\sup_{s \leq t} \mathbb{E}|Y_s|^{2m} < \infty, \tag{3.9}$$

for any $t \geq 0$ and $m \geq 1$ (this is similar to [26, Problem 5.3.15]). Hence, given the representation of Proposition 3.2, it suffices to show that for some constants $\mathcal{M}_i^n < \infty$ and non-negative integers r_i^n:

$$\left|(B_j^n)^{(i)}(y)\right| \leq \mathcal{M}_i^n(1+|y|)^{r_i^n}, \tag{3.10}$$

for any $i, n \geq 0$ and $0 \leq j \leq n$. This claim can again be shown by induction since, given that it is satisfied for $n-1$ and using the iterative representation for B_j^n in Proposition 3.2,

$$\left|(B_j^n)^{(i)}(y)\right| \leq \sum_{\ell=0}^{i}\binom{i}{\ell}\bigg|\frac{1}{2}\left(\gamma^2\right)^{(\ell)}\left(B_j^{n-1}\right)^{(i-\ell+2)} \\ + (\alpha)^{(\ell)}\left(B_j^{n-1}\right)^{(i-\ell+1)} + \frac{1}{2}\left(\sigma^2\right)^{(\ell)}\left(B_{j-1}^{n-1}\right)^{(i-\ell)}\bigg|,$$

which can be bounded by $\mathcal{M}_i^n(1+|y|)^{r_i^n}$ since, by assumption, σ^2, α, and γ^2 belong to $\mathcal{C}_l^k$, for all $k \geq 0$. □

The previous result covers the Heston model,

$$dU_t = -\frac{1}{2}Y_t dt + \sqrt{Y_t}dW_t^{(1)}, \quad dY_t = \chi(\theta - Y_t)dt + v\sqrt{Y_t}dW_t^{(2)}, \tag{3.11}$$

as well as the exponential Ornstein-Uhlenbeck (OU) model:

$$dU_t = -\frac{1}{2}e^{2Y_t}dt + e^{Y_t}dW_t^{(1)}, \quad dY_t = \chi(\theta - Y_t)dt + v dW_t^{(2)}. \tag{3.12}$$

Both models are used in practice.

Let us now use Corollary 3.3 to obtain a second order expansion for $\mathbb{E}h(U_t)$. Omitting, for the ease of notation, the evaluation of the functions B_j^k at y_0, we can write

$$\mathbb{E}h(U_t) = h(0) + B_1^1\mathcal{L}_1 h(0)t + \left(B_1^2\mathcal{L}_1 h(0) + B_2^2\mathcal{L}_1^2 h(0)\right)t^2 + O(t^3),$$

where

$$B_1^1 = \frac{1}{2}\sigma_0^2, \quad B_1^2 = \frac{\gamma_0^2}{2}\left(\sigma_0\sigma_0'' + (\sigma_0')^2\right) + \alpha_0\sigma_0\sigma_0', \quad B_2^2 = \frac{1}{4}\sigma_0^4. \tag{3.13}$$

Above, we set $\sigma_0 = \sigma(y_0)$, $\sigma_0' = \sigma'(y_0)$, and $\sigma_0'' = \sigma''(y_0)$, with similar notation for the other functions. For the Heston model (3.11), the coefficients in (3.13) are

$$B_1^1 = \frac{y_0}{2}, \quad B_2^2 = \frac{y_0^2}{2}, \quad B_1^2 = \frac{\chi(\theta - y_0)}{2}.$$

Similarly, for the exponential OU model (3.12),

$$B_1^1 = \frac{e^{2y_0}}{2}, \quad B_2^2 = \frac{e^{4y_0}}{4}, \quad B_1^2 = e^{2y_0}\left(v^2 + \chi(\theta - y_0)\right).$$

A general polynomial expansion of transition distributions is as follows.

Theorem 3.4. *With the notations and the conditions of Theorem 3.1 and Corollary 3.3,*

$$\mathbb{P}(Z_t \geq z) = e^{-\lambda_\varepsilon t}\sum_{j=1}^{n}\widehat{a}_j(z)\frac{t^j}{j!} + O_{\varepsilon,z_0}(t^{n+1}), \tag{3.14}$$

where

$$\widehat{a}_j(z) := \sum_{p+q+r=j}\binom{j}{p,q,r}L_\varepsilon^q\left(\sum_{m=0}^{r}B_m^r(y_0)\mathcal{L}_1^m(\widehat{f}_{p,z})\right)(0),$$

setting $\widehat{f}_{0,z}(y) \equiv 0$ and where the summation is over all non-negative integers p, q, r.

Proof. It is enough to plug the expansion (3.7) into the coefficients of the first summation in (B-5) (See the proof of Theorem 3.1 in Appendix B) and rearrange terms using Proposition 3.2. Note that the last integral in (3.7) is bounded for $\widetilde{f}_{k,z} \in C_b^\infty(\mathbb{R})$ from the representation in Proposition 3.2 and the estimates (3.9)-(3.10). □

As a way to illustrate the expansions, note that in the case of constant volatility $(\alpha(y) = \gamma(y) \equiv 0)$,

$$B_k^k(y) \equiv \left(\frac{\sigma_0^2}{2}\right)^k, \quad B_j^k(y) \equiv 0, \quad \forall j \neq k, \quad k \geq 0.$$

Hence,

$$\widehat{a}_j(z) := \sum_{p+q+r=j}\binom{j}{p,q,r}\left(\frac{\sigma_0^2}{2}\right)^r L_\varepsilon^q\left(\mathcal{L}_1^r(\widehat{f}_{p,z})\right)(0).$$

Remark 3.5. (i) Clearly, the expansion (3.14) leads to an expansion of the form:

$$\mathbb{P}(Z_t \geq z) = \sum_{j=1}^{n} \breve{a}_j(z) \frac{t^j}{j!} + O_{\varepsilon,z_0}(t^{n+1}). \tag{3.15}$$

Indeed, by expanding $e^{-\lambda_\varepsilon t}$ in (3.14), we have

$$\breve{a}_k(z) = \sum_{j=1}^{k} \binom{k}{j} \widehat{a}_j(z)(-\lambda_\varepsilon)^{k-j}. \tag{3.16}$$

(ii) The coefficients $\breve{a}_k(z)$ in (3.15) are actually independent of ε (for ε small enough) since they can be defined iteratively as limits of $\mathbb{P}\left(X_t \geq y\right)$ as follows:

$$\breve{a}_1(z) = \lim_{t\to 0} \frac{1}{t} \mathbb{P}\left(Z_t \geq z\right), \quad \frac{\breve{a}_k(z)}{k!} = \lim_{t\to 0} \frac{1}{t^k} \left\{ \mathbb{P}\left(Z_t \geq z\right) - \sum_{j=1}^{k-1} \breve{a}_j(z) \frac{t^j}{j!} \right\}.$$

3.2. **Expansions for the call option price.**

For $z \neq 0$ and $t > 0$, let

$$G_t(z) := \mathbb{E}\left(e^{z+Z_t} - 1\right)_+, \tag{3.17}$$

where Z is the jump-diffusion process given by (1.2) and (1.3). We proceed to derive the small-time expansion of G_t as $t \downarrow 0$. We first consider the out-of-the-money case $z < 0$ from which one can easily derive the in-the-money case $z > 0$ via put-call parity (see Corollary 3.9 below). Throughout this section, we set

$$f(u) = f_z(u) := (e^{z+u} - 1)_+,$$

and we also assume the following uniform boundedness condition: there exists $0 < M < \infty$, such that

$$0 < \sigma(y) \leq M. \tag{3.18}$$

Remark 3.6. Under the uniform boundedness condition (3.18), it is easy to see that $\mathbb{E}e^{cU_t} < \infty$, for some $c > 2$. Then, a proof similar to that given in Corollary 3.3, using the representation of Proposition 3.2, shows that (3.7) is satisfied for $g(u, y) := h(u)$, whenever $h \in C^{2n+2}$ is a subexponential function satisfying (2.7).

The next theorem gives an expansion for the out-of-the-money call option prices in terms of the integro-differential operators L_ε and $\mathcal{L}_1$ defined in (2.1) and (3.8) (its proof is given in Appendix C).

Theorem 3.7. *Let $z_0 < 0$, $n \geq 1$, and $0 < \varepsilon < -z_0/2(n+1) \wedge 1$. Let the dynamics of Z be given by (1.2) and (1.3), and the conditions of both Theorem 3.1 and Corollary 3.3 as well as (3.18) be satisfied. Then there exists $t_0 > 0$ such that, for any $0 < t < t_0$ and $z < z_0$,*

$$G_t(z) = e^{-\lambda_\varepsilon t} \sum_{j=1}^{n} \widehat{b}_j(z) \frac{t^j}{j!} + O_{\varepsilon,z_0}(t^{n+1}), \tag{3.19}$$

where

$$\widehat{b}_j(z) := \sum_{p+q+r=j} \binom{j}{p, q, r} L_\varepsilon^q \left(\sum_{m=0}^{r} B_m^r(y_0) \mathcal{L}_1^m(\widehat{f}_{p,z}) \right)(0), \tag{3.20}$$

with $\widehat{f}_{0,z}(y) \equiv 0$*, and*

$$\widehat{f}_{k,z}(y) := \int_{\mathbb{R}} f_z(y+u)\bar{s}_\varepsilon^{*k}(u)du = \int_{\mathbb{R}} \left(e^{z+y+u}-1\right)_+ \bar{s}_\varepsilon^{*k}(u)du.$$

Remark 3.8. By expanding $e^{-\lambda_\varepsilon t}$ in (3.19), one obtains (1.6) with the following coefficients

$$b_k(z) := \sum_{j=1}^{k} \binom{k}{j} \widehat{b}_j(z)(-\lambda_\varepsilon)^{k-j}. \tag{3.21}$$

As it was the case for the expansions of the tail probabilities (see Remark 3.5-(ii) above), the coefficients $b_k(z)$ can be defined iteratively as certain limits of $G_t(z) = \mathbb{E}\left(e^{z+Z_t}-1\right)_+$, as $t \to 0$, and, therefore, they are independent of ε (for ε small enough).

To deal with the in-the-money case $z > 0$, note that

$$\begin{aligned}\mathbb{E}\left(e^{z+Z_t}-1\right)_+ &= \mathbb{E}(e^{z+Z_t}-1) + (e^{z+Z_t}-1)_- \\ &= e^z - 1 + \mathbb{E}(e^{z+Z_t}-1)_-.\end{aligned}$$

The expansion of $\mathbb{E}(e^{z+Z_t}-1)_-$ with $z > 0$ is similar to that of $\mathbb{E}\left(e^{z+Z_t}-1\right)_+$ with $z < 0$. Therefore, we obtain the following result.

Corollary 3.9. *Let* $z_0 > 0$*,* $n \geq 1$*, and* $0 < \varepsilon < z_0/2(n+1) \wedge 1$*. Under conditions of Theorem 3.1, there exists a* $t_0 > 0$ *such that, for any* $0 < t < t_0$*,* $z > z_0$*,*

$$G_t(z) = e^z - 1 + e^{\lambda_\varepsilon t} \sum_{m=1}^{n} \widetilde{b}_m(z) \frac{t^m}{m!} + O_{\varepsilon,z_0}(t^{n+1}), \tag{3.22}$$

where

$$\widetilde{b}_j(z) := \sum_{i+j+k=m} \binom{m}{i,j,k} L_\varepsilon^i \left(\sum_{l=0}^{i} B_l^i(y_0) \mathcal{L}_1^l \widehat{g}_{k,z} \right)(0)$$

with

$$\widehat{g}_{k,z}(y) := \int_{\mathbb{R}} \left(e^{y+z+u}-1\right)_- \bar{s}_\varepsilon^{*k}(u)du.$$

Remark 3.10. The results of this section provide expansions for the option price of out-of-the-money (OTM) and in-the-money (ITM) options in non-negative integer powers of the time to maturity. However, as stated in the introduction, at-the-money (ATM) option prices are known to be expandable as fractional powers, at least in the first leading term (see [39, Proposition 5], [14, Proposition 4.2], and [31]). A natural question is then to try to understand the reasons for such radically different asymptotic behaviors. The key assumption that allows us to obtain non-negative integer powers is the *smoothness* of the Lévy density s outside the origin. Concretely, if s were discontinuous or were not differentiable at a fixed log-moneyness value z, we would expect that the expansions for the tail probability $\mathbb{P}(Z_t \geq z)$ and for the OTM call option price $\mathbb{E}(e^{k+Z_t}-1)_+$ would exhibit fractional powers. This fact was pointed out in [29] for the tail probabilities and a particular type of pure-Lévy processes. The same is true for $\mathbb{E}(e^{k+Z_t}-1)_+$, in view of its representation as the difference of two tail probabilities (see (14) in [14]). In view of this observation, it is now intuitively clear that ATM option prices typically exhibit fractional leading terms. Indeed, at-the-money option prices $\mathbb{E}(e^{Z_t}-1)_+$

correspond to the log-moneyness $z = 0$ and the Lévy density s, is discontinuous at 0 for an infinite-jump activity Lévy process.

3.3. **Other payoff functions.**

One of the advantages of our approach is that it can be applied to more general payoff functions. Concretely, consider a function of the form:

$$f_z(u) := \varphi(u)\mathbf{1}_{\{u\geq z\}},$$

where $\varphi \in C_b^\infty$. One can easily verify that, under the conditions of Theorem 3.4 and for $z > 0$,

$$\mathbb{E} f_z(Z_t) = e^{-\lambda_\varepsilon t}\sum_{j=1}^{n} \widetilde{a}_j(z)\frac{t^j}{j!} + O_{\varepsilon,z_0}(t^{n+1}), \tag{3.23}$$

where

$$\widetilde{a}_j(z) := \sum_{p+q+r=j}\binom{j}{p,q,r} L_\varepsilon^q\left(\sum_{m=0}^{r} B_m^r(y_0)\mathcal{L}_1^m(\widehat{f}_{p,z})\right)(0),$$

with $\widehat{f}_{0,z}(y) = 0$, and $\widehat{f}_{k,z}(y) := \int_{\mathbb{R}} f_z(y+u)\bar{s}_\varepsilon^{*k}(u)du = \int_{z-y}^{\infty}\varphi(y+u)\bar{s}_\varepsilon^{*k}(u)du$. Indeed, from the proof of Theorem 3.1 (which is the key to Theorem 3.4), the only step that requires some extra care is to justify that

$$\widetilde{f}_{k,z}(y) := \lambda_\varepsilon^{-k}\int_{z-y}^{\infty}\varphi(y+u)\bar{s}_\varepsilon^{*k}(u)du$$

is in C^∞ and that $\sup_y |\widetilde{f}_{k,z}^{(j)}(y)| < \infty$. This is proved by verifying (via induction) that

$$\widetilde{f}_{k,z}^{(j)}(y) = \lambda_\varepsilon^{-k}\int_{z-y}^{\infty}\varphi^{(j)}(y+u)\bar{s}_\varepsilon^{*k}(u)du + \lambda_\varepsilon^{-k}\sum_{i=0}^{j-1}(-1)^i\varphi^{(j-1-i)}(z)\bar{s}_\varepsilon^{*(k-1)} * \bar{s}_\varepsilon^{(i)}(z-y).$$

Similarly, under the stronger conditions of Theorem 3.7, one can easily consider payoff functions of the form

$$f_z(u) := \varphi(u)\mathbf{1}_{\{u\geq -z\}}, \quad (z<0),$$

with $\varphi \in C^\infty$ such that $|\varphi^{(j)}(u)| \leq M_j e^u$ for some constant $M_j < \infty$ and all $j \geq 0$.

3.4. **On the short-time large deviation principle for diffusions.**

Large deviation results of the form (3.3) have recently been developed for different stochastic volatility (SV) models. For instance, for uncorrelated SV models,

[17] shows (3.3) under the following conditions:

(3.24) • The function α is uniformly bounded and Lipschitz continuous in $\mathbb{R}$.

(3.25) • $\exists\, M_2 > M_1 > 0,\ s.t.\ 0 \leq M_1 \leq \sigma(y) \wedge \gamma(y) \leq \sigma(y) \vee \gamma(y) \leq M_2 < \infty.$

(3.26) • $\sigma, \gamma \in C^\infty,$ and $\sigma(y) \to \sigma_\pm,\ \gamma(y) \to \gamma_\pm,$ as $y \to \pm\infty.$

(3.27) • σ and γ are diffeomorphisms with $\sigma' > 0$ and $\gamma' > 0$.

(3.28) • $\exists\, y_c \in \mathbb{R},$ such that $\sigma'' > 0,\ \gamma'' > 0\ for\ y < y_c,\ \sigma'' < 0,\ \gamma'' < 0$ for $y > y_c$ and $\sigma' \vee \gamma' < M < \infty$ for some $M > 0$.

(3.29) • The function $u \mapsto \dfrac{\gamma(\sigma^{-1}(u))}{u}$ is non-increasing.

We refer to [17] for an explicit expression for the rate function I, which is not relevant here. The Heston model (3.11) (even with correlated Wiener processes $W^{(1)}$ and $W^{(2)}$) was also considered in [16] and [18].

4. Particular examples and analysis of the parameter contributions

In this section, we shed some light on the specific forms of the general expansions obtained in the previous section for some specific models. We also analyze the contribution, to the call option prices, of the various parameters of the model.

4.1. **Exponential Lévy models.**

Let us start by considering the exponential Lévy model, which can be seen as a particular case of the jump-diffusion models (1.2) and (1.3) by setting $\alpha(y) \equiv \gamma(y) \equiv 0$ and $\sigma(y) \equiv \sigma$. In this setting, the log-return process Z is a Lévy process with generating triplet (b, σ^2, ν). Then, the following expansion for the out-of-the-money call option price holds true (see also [14] for an alternative derivation). Note that the process Z in this section is the same as the process X in Section 2 and therefore all the notations are transferred accordingly.

Corollary 4.1. *Let $z_0 < 0$, $n \geq 1$, and $0 < \varepsilon < -z_0/2(n+1) \wedge 1$. Let $Z = (Z_t)_{t\geq 0}$ be a Lévy process with triplet (b, σ^2, ν) satisfying (2.3)-(2.4). Then there exists $t_0 > 0$ such that, for any $z < z_0$ and $0 < t < t_0$,*

$$G_t(z) = e^{-\lambda_\varepsilon t} \sum_{j=1}^{n} c_j(z) \frac{t^j}{j!} + O_{\varepsilon, z_0}(t^{n+1}), \tag{4.1}$$

where

$$c_j(z) := \sum_{k=1}^{j} \binom{j}{k} L_\varepsilon^{j-k} \widehat{h}_{k,z}(0),$$

with

$$\widehat{h}_{k,z}(y) := \int_{\mathbb{R}} \left(e^{z+y+u} - 1\right)_+ \bar{s}_\varepsilon^{*k}(u)du.$$

As in Corollary 3.9, we also have the following result for the in-the-money case.

Corollary 4.2. *Let $z_0 > 0$, $n \geq 1$, and $0 < \varepsilon < z_0/(n+1) \wedge 1$. Then, there exists $t_0 > 0$ such that, for any $z > z_0$ and $0 < t < t_0$,*

$$G_t(z) = e^z - 1 + e^{-\lambda_\varepsilon t} \sum_{j=1}^{n} \tilde{c}_j(z) \frac{t^j}{j!} + O_{\varepsilon, z_0}(t^{n+1}), \tag{4.2}$$

where

$$\tilde{c}_j(z) := \sum_{k=1}^{j} \binom{j}{k} L_\varepsilon^{j-k} \tilde{h}_{k,z}(0), \quad \tilde{h}_{k,z}(y) := \int_{\mathbb{R}} \left(e^{z+y+u} - 1\right)_- \bar{s}_\varepsilon^{*k}(u) du.$$

4.2. **Second-order expansions.**

Here are the first two coefficients of (3.14) for $\varepsilon > 0$ small enough:

$$\begin{aligned}
\widehat{a}_1(z) &= B_0^0(y_0)\widehat{f}_{1,z}(0) = \int_{\mathbb{R}} f_z(u)\bar{s}_\varepsilon(u)du = \int_z^\infty s(u)du; \\
\widehat{a}_2(z) &= \underbrace{2L_\varepsilon(\widehat{f}_{1,z})(0)}_{p=1,q=1,r=0} + \underbrace{2B_1^1(y_0)\mathcal{L}_1(\widehat{f}_{1,z})(0)}_{p=1,q=0,r=1} + \underbrace{\widehat{f}_{2,z}(0)}_{p=2,q=0,r=0} \\
&= 2\left(b_\varepsilon s(z) - \int_{\mathbb{R}} \int_0^1 s'(z - \beta u)(1-\beta) d\beta u^2 s_\varepsilon(u) du\right) \\
&\quad - \sigma^2(y_0)(s'(z) + s(z)) + \int_{\mathbb{R}^2} \mathbf{1}_{\{u_1+u_2 \geq z\}} \bar{s}_\varepsilon(u_1)\bar{s}_\varepsilon(u_2) du_1 du_2.
\end{aligned}$$

The corresponding coefficients for (3.19) are obtained as above with $f_z(x) = \mathbf{1}_{\{x \geq z\}}$ replaced by $f_z(y) := (e^{z+y} - 1)_+$ with $z < 0$. Hence, for $\varepsilon > 0$ small enough,

$$\begin{aligned}
\widehat{b}_1(z) &= \int_{\mathbb{R}} \left(e^{z+u} - 1\right)_+ s(u) du; \\
\widehat{b}_2(z) &= \sigma^2(y_0)s(-z) + 2b_\varepsilon \int_{-z}^\infty e^{z+u} s(u) du \\
&\quad + \int_{\mathbb{R}^2} \left(e^{z+u_1+u_2} - 1\right)_+ \bar{s}_\varepsilon(u_1)\bar{s}_\varepsilon(u_2) du_1 du_2 \qquad (4.3) \\
&\quad + 2\int_{\mathbb{R}} \int_0^1 (1-\beta) \left(\int_{-z-\beta u}^\infty e^{z-\beta u+w} s(w) dw + s(-z-\beta u)\right) d\beta u^2 s_\varepsilon(u) du.
\end{aligned}$$

In the previous expressions one can substitute $c_\varepsilon(y)$ and $\bar{c}_\varepsilon(y)$ by $\mathbf{1}_{0<|y|<\varepsilon}$ and $\mathbf{1}_{|y|\geq\varepsilon}$, respectively.

Remark 4.3. Combining (1.6), (1.7), and the expression for $\hat{b}_1(z) = b_1(z)$ above gives the following expansion for the price function of the out-of-the-money call option near the expiration T:

$$\begin{aligned}
C(t,s) &= Ke^{-r(T-t)} G_{T-t}(\ln s - \ln K) \\
&= (T-t) \int_{\mathbb{R}} (se^u - K)_+ s(u) du + O_{\varepsilon, \ln(s/K)}\left((T-t)^2\right). \qquad (4.4)
\end{aligned}$$

In particular, the leading term for out-of-the-money call option prices is "dominated" by the jump component of the model regardless of the underlying continuous stochastic volatility component U.

Remark 4.4. Given that in (3.13), $B_0^0(y_0)$ and $B_1^1(y_0)$ depend only on $\sigma(y_0)$, it is interesting to note that the first two coefficients in our expansions (3.14) and (3.19) coincide with the coefficients corresponding to an exponential Lévy model with volatility $\sigma = \sigma(y_0)$. In fact, the initial values of α and γ begin to appear with the coefficients $\widehat{a}_3(z)$ and $\widehat{b}_3(z)$ through the coefficients B_1^2 in (3.13).

4.3. **Parameters contributions.**

For simplicity, let $S_0 = 1$ and $r = 0$. The first term on the right-hand side of (4.3) indicates the main contribution of the spot volatility $\sigma(y_0)$ in the call price. Indeed, recalling (1.8) and the subsequent interpretation of $-z$ as the log-moneyness κ, one can say that the spot volatility has the effect of increasing the call price by

$$\sigma^2(y_0)e^{\kappa}s(\kappa)\frac{t^2}{2}\left(1+o(1)\right). \tag{4.5}$$

This observation was first pointed out in [14] for an exponential Lévy model.

As explained in Remark 4.4, the specific information present in the stochastic volatility model manifests itself up to an $O(t^3)$ term. Indeed, by analyzing each of the terms of the expansion for $\widehat{b}_3$ given in (3.20), the spot drift $\alpha(y_0)$ and the diffusion component $\gamma(y_0)$ of the underlying stochastic volatility factor Y are only felt through the term $B_1^2(y_0)e^{\kappa}s(\kappa)t^3/2$, which, in light of (3.13), takes the form:

$$\left(\frac{\gamma^2(y_0)}{2}\left(\sigma(y_0)\sigma''(y_0)+(\sigma'(y_0))^2\right)+\alpha(y_0)\sigma(y_0)\sigma'(y_0)\right)e^{\kappa}s(\kappa)\,\frac{t^3}{2}. \tag{4.6}$$

For instance, in the Heston model (3.11), $\sigma(y) = \sqrt{y}$ and the effect of the stochastic volatility starts to being felt through the third order term

$$\frac{\chi(\theta - y_0)}{4}e^{\kappa}s(\kappa)\,t^3. \tag{4.7}$$

So, when the volatility is high (low), relative to its average value θ, a mean-reversion speed of χ will decrease (increase) the call price by the amount (4.7). This result is intuitive as the larger the speed is, the faster one would expect the volatility to come back to its average value when this is high, hence resulting in a discount. In a Heston model, the effect of the volatility of volatility v appears up to a fourth degree term of the form

$$v^2 y_0 e^{\kappa}\left(s''(\kappa)-s'(\kappa)\right)\frac{t^4}{4!}.$$

In contrast for the exponential OU model (3.12), the quantity (4.6) simplifies to

$$\frac{e^{2y_0}\left(\chi(\theta-y_0)+v^2\right)}{2}e^{\kappa}s(\kappa)\,t^3$$

and both the speed χ of mean reversion and the volatility of volatility v already appear in the third order term. In spite of appearing in a third-order term, in some situations the contribution of the stochastic volatility model could be significant such as under a fast-mean reverting framework where the value of χ is high. This phenomenon has already been suggested in the literature, partially motivated by empirical evidence (see, e.g., [19]).

By analyzing the terms in (3.20), one can see that in addition to the term (4.6) (which clearly involves the information of σ at y_0), the contribution of the volatility

$\sigma(\cdot)$ to the third-order term $\widehat{b}_3$ also appears in the following term(s):

$$\left(\frac{3}{4}\sigma^4(y_0)\left(s''(\kappa)+s'(\kappa)\right)+\frac{3\sigma^2(y_0)}{2}\left(\bar{s}_\varepsilon^{*2}(\kappa)-2b_\varepsilon s'(\kappa)\right)\right)e^\kappa\frac{t^3}{3!}. \tag{4.8}$$

Identifying the effect of each model component at each level of resolution $t, t^2, \dots,$ might be more enlightening than just explicitly writing down each of the terms. For instance, one can say that under a Heston model with independent Lévy jumps, the first and second-order terms are the same as those of the underlying exponential Lévy model with volatility $\sigma(y_0) = y_0$; while the third order term is the superposition of the corresponding third order term of the underlying exponential Lévy model with volatility $\sigma(y_0) = y_0$ and the term (4.7). From here, we can write the third order term of the underlying exponential Lévy model using the expansion of Theorem 3.7 or using the following representation obtained in [14] (see the proof of Proposition 2.2 therein):

$$\mathbb{E}\left(e^{\widetilde{X}_t}-e^\kappa\right)_+=\mathbb{P}(X_t^*\geq\kappa)-e^\kappa\mathbb{P}(\widetilde{X}_t\geq\kappa). \tag{4.9}$$

Here, $\widetilde{X}$ represents a Lévy process with triplet $(\tilde{b}, \sigma^2(y_0), \nu)$, while X^* represents a Lévy process with triplet $(b^*, \sigma^2(y_0), \nu^*)$, where

$$\tilde{b}=-\frac{1}{2}\sigma^2(y_0)-\int_{-\infty}^{\infty}(e^x-1-x1_{|x|\leq1})\nu(dx), \quad \nu^*(dx)=e^x\nu(dx), \tag{4.10}$$

$$b^*=\frac{1}{2}\sigma^2(y_0)-\int_{-\infty}^{\infty}(e^x-1-x1_{|x|\leq1})\nu(dx)+\int_{|x|\leq1}x\left(e^x-1\right)\nu(dx). \tag{4.11}$$

Next, let $d_j(\kappa; b, \sigma, \nu)$ be the j^{th}-order term in the tail expansion of a Lévy process X with triplet (b, σ^2, ν), i.e., the d_j's are such that

$$\mathbb{P}(X_t\geq\kappa)=\sum_{j=1}^{n}d_j(\kappa;b,\sigma,\nu)\frac{t^j}{j!}+O(t^{n+1}).$$

Then, the third-order term of the option price (4.9) takes the form

$$\left(d_3(\kappa;b^*,\sigma(y_0),\nu^*)-e^\kappa d_3(\kappa;\tilde{b},\sigma(y_0),\nu)\right)\frac{t^3}{3!},$$

which in turn allows us to write the third-order term of the model (1.1)-(1.3) with the Heston specification (3.11) as

$$\left(d_3(\kappa;b^*,\sigma(y_0),\nu^*)-e^\kappa d_3(\kappa;\tilde{b},\sigma(y_0),\nu)+\frac{3\chi(\theta-y_0)}{2}e^\kappa s(\kappa)\right)\frac{t^3}{3!}.$$

We can further disentangle the volatility effect from the second and third order terms by using (4.5) and (4.8). Specifically, under the Heston model (where $\sigma(y) = \sqrt{y}$), the second-order term admits the representation

$$\left(d_2(\kappa;b^*,0,\nu^*)-e^\kappa d_2(\kappa;\tilde{b},0,\nu)+y_0e^\kappa s(\kappa)\right)\frac{t^2}{2}, \tag{4.12}$$

while the third order term can be written as

$$\begin{aligned}\frac{t^3}{3!}\Bigg(& d_3(\kappa; b^*, 0, \nu^*) - e^{\kappa} d_3(\kappa; \tilde{b}, 0, \nu) \\ & + \frac{3\chi(\theta - y_0)}{2} e^{\kappa} s(\kappa) + \frac{3}{4} y_0^2 \left(s''(\kappa) + s'(\kappa)\right) e^{\kappa} \\ & + \frac{3}{2} y_0 \left(\bar{s}_{\varepsilon}^{*2}(\kappa) - 2b_{\varepsilon} s'(\kappa) - 2s(\kappa)\lambda_{\varepsilon}\right) e^{\kappa} \Bigg). \end{aligned} \tag{4.13}$$

In these expressions, b^* and $\tilde{b}$ are computed as in (4.10)-(4.11) but setting $\sigma(y_0) = 0$. Note also that the term $-s(\kappa)e^{\kappa}\lambda_{\varepsilon}t^3/2$ in (4.13) comes from expanding $e^{-\lambda_{\varepsilon}t}$ in the second-order term $e^{-\lambda_{\varepsilon}t}\widehat{b}_2(z)t^2/2$, and isolating the contribution of $\sigma(y_0)$ in the latter term (see (4.5)). The expressions for $d_3(\kappa; b^*, 0, \nu^*)$ and $d_3(\kappa; \tilde{b}, 0, \nu)$ can easily be written from the expressions in [13] (see Theorem 4.3 therein) or from Theorem 3.4 above.

Alternatively, one could also use (4.12) and (4.13) to develop the second and third order SV corrections to call option prices of pure-jump Lévy models. Concretely, the following expansions hold for the model (1.1)-(1.3) with the Heston SV specification (3.11):

$$\begin{aligned}\mathbb{E}\left(S_t - e^{\kappa}\right)_+ &= \mathbb{E}\left(e^{X_t} - e^{\kappa}\right)_+ + \sigma^2(y_0) e^{\kappa} s(\kappa)\frac{t^2}{2} + O(t^3), \\ &= \mathbb{E}\left(e^{X_t} - e^{\kappa}\right)_+ + \sigma^2(y_0) e^{\kappa} s(\kappa)\frac{t^2}{2} \\ &\quad + \Bigg(\frac{3\chi(\theta - y_0)}{2} e^{\kappa} s(\kappa) + \frac{3}{4} y_0^2 \left(s''(\kappa) + s'(\kappa)\right) e^{\kappa} \\ &\qquad + \frac{3}{2} y_0 \left(\bar{s}_{\varepsilon}^{*2}(\kappa) - 2b_{\varepsilon} s'(\kappa) - 2s(\kappa)\lambda_{\varepsilon}\right) e^{\kappa}\Bigg)\frac{t^3}{3!} + O(t^4).\end{aligned}$$

Above, X denotes the pure-jump Lévy model underlying S, i.e., X is a Lévy process with Lévy triplet $(b, 0, \nu)$, with b chosen as in (1.4). The call option prices $\mathbb{E}\left(e^{X_t} - e^{\kappa}\right)_+$ can be computed using Fourier inversion formulas or, in some particular cases, via analytic closed form formulas.

5. Asymptotics of the implied volatility

Using the leading term of the time-t price for the out-of-the-money call option as computed in the previous section, we now obtain the asymptotic behavior of the implied volatility $\hat{\sigma}(t; s)$ near T. This is defined implicitly by the equation

$$C(t, s) = C_{BS}(t, s; \hat{\sigma}(t; s), r), \tag{5.1}$$

where $C_{BS}(t, s; \sigma, r)$ is the classical time-t Black-Scholes call-option price corresponding to an interest rate r, a volatility σ, and time-t spot price s. First, we need the following well-known result (see, e.g., Lemma 2.5 in [20]).

Lemma 5.1. *Let $C_{BS}(t,s;\sigma,r)$ be the classical Black-Scholes call price function. Then, as $t \uparrow T$,*

$$C_{BS}(t,s;\sigma,r) = \frac{1}{\sqrt{2\pi}} \frac{K\sigma^3(T-t)^{3/2}}{(\ln K - \ln s)^2} \exp\left(-\frac{(\ln K - \ln s)^2}{2\sigma^2(T-t)}\right) \times \exp\left(-\frac{\ln K - \ln s}{2} + \frac{r(\ln K - \ln s)}{\sigma^2}\right) + R(t,s;\sigma,r). \tag{5.2}$$

The remainder satisfies

$$|R(t,s;\sigma,r)| \le M(T-t)^{5/2} \exp\left(-\frac{(\ln K - \ln s)^2}{2\sigma^2(T-t)}\right), \tag{5.3}$$

where $M = M(s,\sigma,r,K)$ is uniformly bounded if all the indicated parameters vary in a bounded region.

The next result gives the asymptotic behavior of $\hat{\sigma}(t,s)$. This has already been obtained for a pure-Lévy processes (see, e.g., [39] and [14]) and is presented here for the sake of completeness.

Proposition 5.2. *Let the dynamics of Z be given by (1.2) and (1.3), and the conditions of both Theorem 3.1 and Corollary 3.3 as well as (3.18) be satisfied. Let $\hat{\sigma}(t;s)$ be the implied volatility when the time-t stock price S_t is s. Then, as $t \uparrow T$,*

$$\hat{\sigma}^2(t;s) \sim \frac{(\ln K - \ln s)^2}{-2(T-t)\ln(T-t)}. \tag{5.4}$$

Proof. Let $\kappa := \ln K/s$ be the log-moneyness and let $\tau := T-t$ be the time-to-maturity, where, for simplicity, we write $\hat{\sigma}(t)$ instead of $\hat{\sigma}(t;s)$. Using the leading terms in (4.4) and (5.2), we obtain that as $t \uparrow T$:

$$\tau u(s,K) \sim v(s,K)\hat{\sigma}^3(t)\tau^{3/2} \exp\left(-\frac{\kappa^2}{2\hat{\sigma}^2(t)\tau} + \frac{r\kappa}{\hat{\sigma}^2(t)}\right), \tag{5.5}$$

where

$$u(s,K) = \int_{\mathbb{R}} (se^u - K)_+ \, s(u)du, \qquad v(s,K) = \frac{1}{\sqrt{2\pi}} \frac{K}{\kappa^2} e^{-\frac{\kappa}{2}}.$$

Assume that $\limsup_{t\uparrow T} \hat{\sigma}(t)\tau^{1/2} = c \in (0,+\infty)$, then $\limsup_{t\uparrow T} \hat{\sigma}(t) = +\infty$ and, thus,

$$\limsup_{t\uparrow T} \left(\hat{\sigma}(t)\tau^{1/2}\right)^3 \exp\left(-\frac{\kappa^2}{2\hat{\sigma}^2(t)\tau} + \frac{r\kappa}{\hat{\sigma}^2(t)}\right) = c^3 \exp\left(-\frac{\kappa^2}{2c^2}\right) \neq 0.$$

So the right hand side of (5.5) does not converge to 0 while the left hand side does, which is clearly a contradiction. Now if $\limsup_{t\uparrow T} \hat{\sigma}(t)\tau^{1/2} = +\infty$, then $\limsup_{t\uparrow T} \hat{\sigma}(t) = +\infty$ and, thus,

$$\limsup_{t\uparrow T} \left(\hat{\sigma}(t)\tau^{1/2}\right)^3 \exp\left(-\frac{\kappa^2}{2\hat{\sigma}^2(t)\tau} + \frac{r\kappa}{\hat{\sigma}^2(t)}\right) = +\infty.$$

Again we obtain the same contradiction. Therefore, we have $\limsup_{t\uparrow T} \hat{\sigma}(t)\tau^{1/2} = 0$. Then, (5.5) can now be equivalently written as

$$\exp\left(-\frac{\kappa^2}{2\hat{\sigma}^2(t)\tau} + \frac{r\kappa}{\hat{\sigma}^2(t)} + 3\ln\left(\hat{\sigma}(t)\tau^{1/2}\right) - \ln\tau\right) \sim \frac{u(s,K)}{v(s,K)}.$$

Hence, as $t \uparrow T$,

$$\lim_{t\uparrow T}\left(-\frac{\kappa^2}{2\hat{\sigma}^2(t)\tau}+\frac{r\kappa}{\hat{\sigma}^2(t)}+3\ln\left(\hat{\sigma}(t)\tau^{1/2}\right)-\ln\tau-\ln\frac{u(s,K)}{v(s,K)}\right)=0$$

Finally, since $\lim_{t\uparrow T}\hat{\sigma}^2(t)\tau\ln\left(\hat{\sigma}(t)\tau^{1/2}\right)=0$, we obtain (5.4). □

Remark 5.3.

(1) As seen in Proposition 5.2, the leading order term of the implied volatility near expiration is "model free", i.e., it does not depend on any of the model parameters.
(2) As discussed in Remark 4.4, the second order expansion of OTM call option prices coincides with that of a purely exponential Lévy model with volatility parameter σ equal to the spot volatility $\sigma(Y_t)$. Thus, the second-order expansion for the implied volatility given in [14] applies:

$$\hat{\sigma}^2(t;s)=v_0(\tau;\kappa)\left(1+v_1(\tau;\kappa)\,+\,o\left(\frac{1}{\log\frac{1}{\tau}}\right)\right),\qquad(\tau\to 0),$$

where κ and τ denote respectively the spot log-moneyness $\kappa:=\log K/s$ and time-to-maturity $\tau:=T-t$, while v_0 and v_1 are given by

$$v_0(\tau;\kappa)=\frac{\frac{1}{2}\kappa^2}{-\tau\log\tau}\,,$$
$$v_1(\tau;\kappa)=\frac{1}{\log(\frac{1}{\tau})}\log\left(\frac{4\sqrt{\pi}e^{-\kappa/2}}{\kappa}\int(e^u-e^\kappa)_+s(u)du\log^{3/2}\left(\frac{1}{\tau}\right)\right).$$

(3) In the very recent manuscript [22], the authors give a blueprint to generate expansions of arbitrary order for the implied volatility $\hat{\sigma}^2(t;s)$. Interestingly, such expansions are determined exclusively by the leading order term of the option price expansion, meaning that the stochastic volatility correction is not relevant for implied volatility[4].

6. Small-time expansions for the Lévy transition densities

In this part, we revisit the important problem of finding small-time expansions for the transition densities of Lévy processes. This problem has recently been considered in [37] and also in [13]. As in Section 2.1, we consider a general Lévy process X with Lévy triplet (b,σ^2,ν). It is well-known that under general conditions (see, e.g, [28] and [34]):

$$\lim_{t\to 0}\frac{1}{t}f_t(x)=s(x),\quad(x\neq 0),\tag{6.1}$$

where f_t denotes the probability density of X_t and s is the Lévy density of ν (both densities are assumed to exist). In many applications, the following uniform convergence result is more desirable

$$\lim_{t\to 0}\sup_{|x|\geq\eta}\left|\frac{1}{t}f_t(x)-s(x)\right|=0,\tag{6.2}$$

[4]We thank an anonymous referee for bringing our attention to this manuscript.

for a fixed $\eta > 0$. The limit (6.2) is related to the general expansions of the transition densities:

$$f_t(x) = \sum_{k=1}^{n} \frac{a_k(x)}{k!} t^k + t^{n+1} O_\eta(1), \tag{6.3}$$

which is valid for any $|x| \geq \eta$ and $0 < t < t_0$, with t_0 possibly depending on the given $\eta > 0$ and $n \geq 0$. Above, and as in (3.4), $O_\eta(1)$ denotes a function of x and t such that

$$\sup_{0<t\leq t_0} \sup_{|x|\geq\eta} |O_\eta(1)| < \infty.$$

Note that (6.2) follows from (6.3) when $n = 1$ and $a_k(x) = s(x)$.

The expansion (6.2) was first proposed in [37] building on results of [28], where the pointwise convergence in (6.1) was obtained. In both papers, the standing assumptions on the Lévy density s of the Lévy process X are:

$$\textbf{(i)}\ \text{There exists } 0 < \alpha < 2, \text{ such that } \liminf_{\eta\to 0} \eta^{\alpha-2} \int_{-\eta}^{\eta} z^2 s(z)dz > 0; \tag{6.4}$$

$$\textbf{(ii)}\ s \in C^\infty(\mathbb{R} \setminus \{0\}); \tag{6.5}$$

$$\textbf{(iii)}\ \int_{|z|\geq\eta} \frac{|s'(z)|^2}{s(z)} dz < \infty, \quad \text{for all } \eta > 0; \tag{6.6}$$

$$\textbf{(iv)}\ \text{There exists } h \in C^\infty \text{ such that } h(z) = O(z^2) \ \ (z \to 0), \tag{6.7}$$

$$h(z) > 0 \text{ if } s(z) > 0, \ \text{ and } \int_{|z|\leq 1} \left| \frac{d}{dz} h(z)s(z) \right|^2 \frac{1}{s(z)} dz < \infty.$$

Condition (6.4) is used to conclude the existence of a C^∞ transition density f_t (see [38, Chapter 5]), while (6.5)-(6.7) are needed to establish an estimate for the transition density using Malliavin Calculus. However, the method of proof of [37] is not convincing and can only yield the first order expansion in (6.3) (see the introduction of [13] for more details). Recently, [13] obtained (6.3) under the following two hypotheses:

$$\gamma_{\eta,k} := \sup_{|x|\geq\eta} |s^{(k)}(x)| < \infty, \tag{6.8}$$

$$\limsup_{t\searrow 0} \sup_{|x|\geq\eta} |f_t^{(k)}(x)| < \infty, \qquad \text{for all } k \geq 0 \text{ and for all } \eta > 0. \tag{6.9}$$

Condition (6.8) is quite mild but condition (6.9) could be hard to prove in general since closed-form expressions of the densities f_t are lacking. Nevertheless [13] shows that condition (6.9) is satisfied by, e.g., the CGMY model of [9] or Koponen [27] and by other types of tempered stable Lévy processes (as defined in [35]).

In this section, we show that (6.9) is not needed to obtain (6.3). See Appendix C for the proof of the following result.

Theorem 6.1. *Let $\eta > 0$ and $n \geq 1$, and let the conditions (6.4)-(6.8) be satisfied. Then, (6.3) holds true for all $0 < t \leq 1$ and $|x| \geq \eta$. Moreover, there exists $\varepsilon_0(\eta, n) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the coefficients a_k admit the following* *representation (which is moreover independent of ε for any $0<\varepsilon<\varepsilon_0$):*

$$a_k(x) := \sum_{j=1}^{k} \binom{k}{j} (-\lambda_\varepsilon)^{k-j} \sum_{i=1}^{j} \binom{j}{i} L_\varepsilon^{j-i} \hat{s}_{i,x}(0), \tag{6.10}$$

where $\hat{s}_{i,x}(u) := \bar{s}_\varepsilon^{*i}(x-u)$.

Remark 6.2. Combining the proofs of Theorem 3.1 and Theorem 6.1, it is possible to obtain a small-time expansion for the jump-diffusion model (1.2)-(1.3) assuming, for instance, that the stochastic volatility model admits a density function d_t satisfying the small-time estimate:

$$\sup_{|x|\geq \eta} d_t(x) \leq M_{p,\eta} t^p,$$

for any $p \geq 1$ and $0<t<t_0(p,\eta)$ and some constant $M_{p,\eta}<\infty$.

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Appendix A. Proof of Lemma 2.1

Let us show (2.6) for $n=1$ (the other cases are easily obtained by induction). First, applying Itô's lemma ([25, Theorem I.4.56]),

$$\begin{aligned} g(X_t) = g(0) + \int_0^t Lg(X_u)du + \sigma \int_0^t g'(X_u)dW_u \\ + \int_0^t \int_{\mathbb{R}} (g(X_{u^-}+z) - g(X_{u^-}))\bar{\mu}(du,dz), \end{aligned}$$

where L is given by (1.10). One can easily check that $Lg(x)$ is well-defined in light of the continuity of $g^{(i)}$ and (2.7). Indeed, there exist constants $M_i, i=0,1,2$, such that $|g^{(i)}(x)| \leq M_i e^{\frac{c}{2}|x|}$, for all x, and thus,

$$\begin{aligned} \left| \int_{|z|\geq 1} g(x+z)\nu(dz) \right| &\leq M_0 \int_{|z|\geq 1} e^{\frac{c}{2}|z|}\nu(dz) e^{\frac{c}{2}|x|}, \\ \left| \int_{|z|\leq 1} (g(x+z) - g(x) - g'(x)z)\nu(dz) \right| &\leq M_2 e^{\frac{c}{2}} \int_{|z|\leq 1} z^2 \nu(dz) e^{\frac{c}{2}|x|}. \end{aligned}$$

Next, we show that the last two terms of the expansion of $g(X_t)$ above are true martingales. Indeed, it suffices that

$$\mathbb{E}\int_0^t |g'(X_u)|^2 du < \infty, \tag{A-1}$$

$$\mathbb{E}\int_0^t \int_{|z|>1} |g(X_u+z) - g(X_u)| \nu(dz)du < \infty, \tag{A-2}$$

$$\mathbb{E}\int_0^t \int_{|z|\leq 1} |g(X_u+z) - g(X_u)|^2 \nu(dz)du < \infty. \tag{A-3}$$

Using (2.7) and the continuity of g', there exists a constant $M>0$ such that

$$\mathbb{E}\int_0^t |g'(X_u)|^2 du \leq M \int_0^t \mathbb{E} e^{c|X_u|} du \leq M \int_0^t \mathbb{E} e^{cX_u} du + \int_0^t \mathbb{E} e^{-cX_u} du < \infty,$$

for any $t \geq 0$. Similarly, setting $\bar{B} = \{z : |z| > 1\}$, (A-2) is satisfied since

$$\mathbb{E}\int_0^t \int_{\bar{B}} |g(X_u + z) - g(X_u)|\, \nu(dz)du \leq \mathbb{E}\int_0^t \int_{\bar{B}} \left|\int_0^z g'(X_u + w)dw\right| \nu(dz)du$$
$$\leq M\int_0^t \mathbb{E}e^{c|X_u|}du \int_{\bar{B}}\int_0^{|z|} e^{cw}dw\nu(dz) < \infty.$$

Also, setting $B = \{z : |z| \leq 1\}$,

$$\mathbb{E}\int_0^t \int_B |g(X_u + z) - g(X_u)|^2\, \nu(dz)du \leq \mathbb{E}\int_0^t \int_B \int_0^1 |g'(X_u + z\beta)|^2 d\beta z^2\nu(dz)du$$
$$\leq \int_0^t \mathbb{E}e^{c|X_u|}du \int_B \int_0^1 e^{c|z|\beta}d\beta z^2\nu(dz) < \infty.$$

We then have that

$$\mathbb{E}g(X_t) = g(0) + \mathbb{E}\int_0^t Lg(X_u)du,$$

which leads to (2.5), provided $\int_0^t \mathbb{E}\,|Lg(X_u)|\, du < \infty$. The latter is proved using (2.7) and arguments as above.

In order to obtain (2.6) for $n = 1$ by iterating (2.5), we need to show that for any C^4 function g satisfying (2.7),

$$\limsup_{|y|\to\infty}\; e^{-\frac{c}{2}|y|}|(Lg)^{(i)}(y)| < \infty, \tag{A-4}$$

for $i = 0, 1, 2$. To this end, we first note that

$$(Lg)^{(i)}(y) = bg^{(i+1)}(y) + \frac{\sigma^2}{2}g^{(i+2)}(y) + \int_{\mathbb{R}}(g^{(i)}(y+z) - g^{(i)}(y) - zg^{(i+1)}(y)\mathbf{1}_{|z|\leq 1})\nu(dz)$$

for $i = 0, 1, 2$. Hence, it is sufficient to show (A-4) when $i = 0$. But,

$$e^{-\frac{c}{2}|y|}|Lg(y)| \leq be^{-\frac{c}{2}|y|}|g'(y)| + \frac{\sigma^2}{2}e^{-\frac{c}{2}|y|}|g''(y)| \tag{A-5}$$

$$+\, e^{-\frac{c}{2}|y|}\int_{|z|>1}|g(y+z) - g(y)|\nu(dz) \tag{A-6}$$

$$+\, e^{-\frac{c}{2}|y|}\int_{|z|\leq 1}|g(y+z) - g(y) - zg'(y)|\nu(dz). \tag{A-7}$$

The limits of each term of the right-hand terms in (A-5) are trivially finite as $|y| \to \infty$ by the assumption (2.7). For the term in (A-6), again by the assumption (2.7) and the continuity of $g^{(i)}$, there exists $M > 0$ such that,

$$|g^{(i)}(y)| \leq Me^{\frac{c}{2}|y|}, \quad i = 0, 1, 2.$$

It then follows that

$$e^{-\frac{c}{2}|y|}\int_{|z|>1}|g(y+z) - g(y)|\nu(dz) = e^{-\frac{c}{2}|y|}\int_{|z|>1}\Big|\int_0^z g'(y+w)\Big|dw\Big|\nu(dz)$$
$$\leq M\int_{|z|>1}\Big(\int_0^{|z|} e^{\frac{c}{2}w}dw\Big)\nu(dz)$$
$$= M\int_{|z|>1} e^{\frac{c}{2}|z|}\nu(dz) < \infty,$$

which immediately implies that

$$\limsup_{|y|\to\infty} e^{-\frac{c}{2}|y|}\int_{|z|>1}|g(y+z)-g(y)|\nu(dz)<\infty. \tag{A-8}$$

Similarly, we can show that the limit as $|y|\to\infty$ of (A-7) is finite. Therefore, we can iterate (2.5) to obtain (2.6) for $n=1$. □

Appendix B. Proof of Theorem 3.1

We analyze each term on the right-hand side of the expansion of $\mathbb{E}f(Z_t)$ given in (3.1)-(3.2):

(1) For any $z\geq z_0$, we have

$$\mathbb{E}f_z\left(U_t+X_t^{\varepsilon}\right)=\mathbb{P}(U_t+X_t^{\varepsilon}\geq z)\leq \mathbb{P}(U_t\geq z/2)+\mathbb{P}(X_t^{\varepsilon}\geq z/2). \tag{B-1}$$

By our assumption (3.3), there exists $t_0(z_0)>0$ such that for any $0<t\leq t_0$, $z\geq z_0>0$,

$$\mathbb{P}(U_t\geq z/2)\leq \mathbb{P}(U_t\geq z_0/2)\leq \exp\left(-\frac{d(z_0/2)^2}{4t}\right), \tag{B-2}$$

which can be seen to be $O_{z_0}(t^{n+1})$. Also, the second term on the right-hand-side of (B-1) is $O_{\varepsilon,z_0}(t^{n+1})$ in light of (2.2) by taking $a:=(n+1)/z_0$ and since $0<\varepsilon<z_0/(n+1)\wedge 1$.

(2) The second term in (3.1) is also $O_{\varepsilon,z_0}(t^{n+1})$ since $f\leq 1$ and clearly

$$e^{-\lambda_\varepsilon t}\sum_{k=n+1}^{\infty}(\lambda_\varepsilon t)^k/k!\leq(\lambda_\varepsilon t)^{n+1}=O(t^{n+1}).$$

(3) We proceed to deal with the terms in (3.2). By the independence of U and X,

$$\mathbb{E}f_z\left(U_t+X_t^{\varepsilon}+\sum_{i=1}^{k}\xi_i\right)=\mathbb{E}\widetilde{f}_{k,z}\left(U_t+X_t^{\varepsilon}\right)=\mathbb{E}\breve{f}_{k,z,t}(X_t^{\varepsilon}), \tag{B-3}$$

where

$$\widetilde{f}_{k,z}(y):=(\lambda_\varepsilon)^{-k}\int_{z-y}^{\infty}\bar{s}_{\varepsilon}^{*k}(u)du \quad \text{and} \quad \breve{f}_{k,z,t}(y):=\mathbb{E}\widetilde{f}_{k,z}\left(U_t+y\right).$$

In particular, by the assumption (2.3),

$$\widetilde{f}_{k,z}^{(j)}(y)=(\lambda_\varepsilon)^{-k}(-1)^{j-1}\bar{s}_{\varepsilon}^{*(k-1)}*\bar{s}_{\varepsilon}^{(j-1)}(z-y),$$
$$\sup_{y,z}\left|\widetilde{f}_{k,z}^{(j)}(y)\right|\leq\lambda_\varepsilon^{-1}\|\bar{s}_{\varepsilon}^{(j-1)}\|_\infty\leq\lambda_\varepsilon^{-1}\max_{0\leq i\leq j-1}\gamma_{i,\varepsilon/2}:=\Gamma_\varepsilon<\infty.$$

It follows that $\breve{f}_{k,z,t}\in C_b^{\infty}(\mathbb{R})$ and moreover,

$$\breve{f}_{k,z,t}^{(j)}(y)=\mathbb{E}\widetilde{f}_{k,z}^{(j)}\left(U_t+y\right), \text{ and } \sup_{z,y}\left|\breve{f}_{k,z,t}^{(j)}(y)\right|\leq\Gamma_\varepsilon, \text{ for any } j\geq 0. \tag{B-4}$$

We can thus apply the iterated formula (2.6) to get
(B-5)

$$\mathbb{E}\breve{f}_{k,z,t}(X_t^{\varepsilon})=\sum_{i=0}^{n-k}\frac{t^i}{i!}L_\varepsilon^i\breve{f}_{k,z,t}(0)+\frac{t^{n-k+1}}{(n-k)!}\int_0^1(1-\alpha)^{n-k}\mathbb{E}\{L_\varepsilon^{n-k+1}\breve{f}_{k,z,t}(X_{\alpha t}^{\varepsilon})\}d\alpha.$$

It follows from the representation in Lemma 2.2 and (B-4) that

$$\sup_z \int_0^1 (1-\alpha)^{n-k} \mathbb{E}\big(L_\varepsilon^{n-k+1} \breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)\big) d\alpha < \infty,$$

and thus the second term on the right hand side of (B-5) is $O_{\varepsilon,z_0}(t^{n-k+1})$.

(4) Combining (3.1), (3.2) and (B-5), we obtain

$$\begin{aligned}
\mathbb{E}f(Z_t) &= e^{-\lambda_\varepsilon t} \sum_{k=1}^n \frac{(\lambda_\varepsilon t)^k}{k!} \mathbb{E}\breve{f}_{k,z,t}(X_t^\varepsilon) + O_{\varepsilon,z_0}(t^{n+1}) \\
&= e^{-\lambda_\varepsilon t} \sum_{k=1}^n \frac{(\lambda_\varepsilon t)^k}{k!} \sum_{i=0}^{n-k} \frac{t^i}{i!} L_\varepsilon^i \breve{f}_{k,z,t}(0) + O_{\varepsilon,z_0}(t^{n+1}) \\
&= e^{-\lambda_\varepsilon t} \sum_{j=1}^n \frac{t^j}{j!} \sum_{k=1}^j \binom{j}{k} \lambda_\varepsilon^k L_\varepsilon^{j-k} \breve{f}_{k,z,t}(0) + O_{\varepsilon,z_0}(t^{n+1}).
\end{aligned}$$

Using again the representation in Lemma 2.2 and (B-4), it follows that

$$L_\varepsilon^{j-k} \breve{f}_{k,z,t}(x) = L_\varepsilon^k \left[\mathbb{E}\widetilde{f}_{k,z}(U_t + \cdot)\right](x) = \lambda_\varepsilon^{-k} L_\varepsilon^k \left[\mathbb{E}\widehat{f}_{k,z}(U_t + \cdot)\right](x),$$

and (3.5) is obtained. □

Appendix C. Proof of Theorem 3.7

We analyze each term in (3.1) and (3.2) through the following steps:

(1) For $z \le z_0 < 0$,

$$\begin{aligned}
\text{(C-1)} \quad \mathbb{E}f_z(U_t + X_t^\varepsilon) &= \mathbb{E}\left(e^{z+U_t+X_t^\varepsilon} - 1\right)_+ \le \mathbb{E}\left(e^{U_t+X_t^\varepsilon} \mathbf{1}_{\{U_t+X_t^\varepsilon \ge -z\}}\right) \\
&\le \left(\mathbb{E}e^{2U_t+2X_t^\varepsilon} \mathbb{P}(U_t + X_t^\varepsilon \ge -z)\right)^{1/2} \\
&\le \left(\mathbb{E}e^{2U_t}\mathbb{E}e^{2X_t^\varepsilon}\right)^{1/2} \Big(\mathbb{P}(U_t \ge -z/2) + \mathbb{P}(X_t^\varepsilon \ge -z/2)\Big)^{1/2}, \\
&= e^{t\psi(2)/2} \left(\mathbb{E}e^{2U_t}\right)^{1/2} \Big(\mathbb{P}(U_t \ge -z/2) + \mathbb{P}(X_t^\varepsilon \ge -z/2)\Big)^{1/2},
\end{aligned}$$

where ψ is the characteristic exponent of X^ε. Since $M_t := e^{U_t}$ satisfies the SDE $dM_t = M_t \sigma(Y_t) dW_t^{(1)}$, and from the Burkhölder-Davis-Gundy inequality,

$$\begin{aligned}
\mathbb{E}e^{2U_t} &= \mathbb{E}\left(1 + \int_0^t M_s \sigma(Y_s) dW_s^{(1)}\right)^2 \\
&\le 2 + 2\mathbb{E}\left(\int_0^t e^{U_s} \sigma(Y_s) dW_s^{(1)}\right)^2 \le 2 + 2M^2\, \mathbb{E}\int_0^t e^{2U_s} ds.
\end{aligned}$$

By Gronwall's inequality,

$$\mathbb{E}e^{2U_t} \le 2e^{2M^2 t} = O_{\varepsilon,z_0}(1).$$

Therefore, the right-hand-side of (C-1) is of order $o_{\varepsilon,z_0}(t^{n+1})$ by (2.2) and (3.3).

(2) The second summation in (3.1) is also $O_{\varepsilon,z_0}(t^{n+1})$ since for any $k \geq n+1$,

$$\mathbb{E} f_z(U_t + X_t^\varepsilon + \sum_{i=1}^k \xi_i) \leq e^z \mathbb{E} e^{U_t} \mathbb{E} e^{X_t^\varepsilon} (\mathbb{E} e^{\xi_1})^k \leq \lambda_\varepsilon^{-k} e^{t\Psi(1)} \left(\int_{\mathbb{R}} e^x \bar{s}_\varepsilon(x) dx \right)^k.$$

(3) To study the summation in (3.2), recall that by the independence of U and X, for any $1 \leq k \leq n$,

$$\mathbb{E} f_z \left(U_t + X_t^\varepsilon + \sum_{i=1}^k \xi_i \right) = \mathbb{E} \widetilde{f}_{k,z} \left(U_t + X_t^\varepsilon \right) = \mathbb{E} \breve{f}_{k,z,t}(X_t^\varepsilon),$$

where

$$\breve{f}_{k,z,t}(x) = \mathbb{E} \widetilde{f}_{k,z} \left(U_t + x \right) \quad \text{ and } \quad \widetilde{f}_{k,z}(x) = \mathbb{E} f_z \left(x + \sum_{i=1}^k \xi_i \right).$$

Let us show that $\tilde{f}_{k,z}$ is C^∞. Indeed, since

$$\widetilde{f}_{k,z}(x) = \lambda_\varepsilon^{-k} \int_{\mathbb{R}^{k-1}} \int_{-\sum_{\ell=2}^k u_\ell - z - x}^{\infty} \left(e^{z+x+\sum_{\ell=1}^k u_\ell} - 1 \right) \bar{s}_\varepsilon(u_1) du_1 \prod_{\ell=2}^k \bar{s}_\varepsilon(u_\ell) du_\ell,$$

and $\bar{s}_\varepsilon \in C_b^\infty$, we have that

$$\begin{aligned} \widetilde{f}'_{k,z}(x) &= \lambda_\varepsilon^{-k} \int_{\mathbb{R}^{k-1}} \int_{-\sum_{\ell=2}^k u_\ell - z - x}^{\infty} e^{z+x+\sum_{\ell=1}^k u_\ell} \bar{s}_\varepsilon(u_1) du_1 \prod_{\ell=2}^k \bar{s}_\varepsilon(u_\ell) du_\ell, \\ \widetilde{f}''_{k,z}(x) &= \lambda_\varepsilon^{-k} \int_{\mathbb{R}^{k-1}} \int_{-\sum_{\ell=2}^k u_\ell - z - x}^{\infty} e^{z+x+\sum_{\ell=1}^k u_\ell} \bar{s}_\varepsilon(u_1) du_1 \prod_{\ell=2}^k \bar{s}_\varepsilon(u_\ell) du_\ell \\ &\quad + \lambda_\varepsilon^{-k} \int_{\mathbb{R}^{k-1}} \bar{s}_\varepsilon \left(-\sum_{\ell=2}^k u_\ell - z - x \right) \prod_{\ell=2}^k \bar{s}_\varepsilon(u_\ell) du_\ell. \end{aligned}$$

Using induction, we see that

$$\begin{aligned} \widetilde{f}^{(i)}_{k,z}(x) &= \lambda_\varepsilon^{-k} \int_{\mathbb{R}} \int_{-\sum_{\ell=2}^k u_\ell - z - x}^{\infty} e^{z+x+\sum_{\ell=1}^k u_\ell} \bar{s}_\varepsilon(u_1) du_1 \prod_{\ell=2}^k \bar{s}_\varepsilon(u_\ell) du_\ell \\ &\quad + \lambda_\varepsilon^{-k} \sum_{j=0}^{i-2} (-1)^j \int_{\mathbb{R}^{k-1}} \bar{s}_\varepsilon^{(j)} \left(-\sum_{\ell=2}^k u_\ell - z - x \right) \prod_{\ell=2}^k \bar{s}_\varepsilon(u_\ell) du_\ell. \end{aligned} \tag{C-2}$$

In view of (2.3), there exists a constant $M_{i,\varepsilon} < \infty$ such that, for any $i \geq 1$,

$$\begin{aligned} \left| \widetilde{f}^{(i)}_{k,z}(U_t + x) \right| &\leq \lambda_\varepsilon^{-k} \int_{\mathbb{R}^k} e^{z+x+\sum_{\ell=1}^k u_\ell} \prod_{\ell=1}^k \bar{s}_\varepsilon(u_\ell) du_\ell \cdot e^{U_t} \\ &\quad + M_{i,\varepsilon} \lambda_\varepsilon^{-k} \sum_{j=0}^{i-2} \int_{\mathbb{R}^{k-1}} \prod_{\ell=2}^k \bar{s}_\varepsilon(u_\ell) du_\ell \max_{0 \leq j \leq i} \gamma_{j,\varepsilon/2}. \end{aligned} \tag{C-3}$$

The right-hand side of (C-3) is integrable since $\mathbb{E} e^{U_t} = 1$. By dominated convergence, we conclude that $\breve{f}_{k,z,t} \in C^\infty(\mathbb{R})$, and also,

$$\breve{f}^{(i)}_{k,z,t}(x) = \mathbb{E} \left[\widetilde{f}^{(i)}_{k,z}(U_t + x) \right], \text{ for all } i \geq 0, \quad \text{and} \quad \limsup_{|x| \to \infty} e^{-\frac{c}{2}|x|} \left| \breve{f}^{(i)}_{k,z,t}(x) \right| < \infty,$$

since $c \geq 2$. Thus, applying (2.6) gives

(C-4)
$$\mathbb{E}\breve{f}_{k,z,t}(X_t^\varepsilon) = \sum_{i=0}^{n-k} \frac{t^i}{i!} L_\varepsilon^i \breve{f}_{k,z,t}(0) + \frac{t^{n-k+1}}{(n-k)!} \int_0^1 (1-\alpha)^{n-k} \mathbb{E}\{L_\varepsilon^{n-k+1} \breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)\} d\alpha.$$

To show that the last integral in (C-4) is bounded, we apply Lemma 2.2 to get that

$$\mathbb{E}\left((L_\varepsilon^{n-k+1}\breve{f}_{k,z,t})(X_{\alpha t}^\varepsilon)\right) = \sum_{\mathbf{k}\in\mathcal{K}_{n-k+1}} \prod_{i=0}^4 b_i^{k_i} \binom{n-k+1}{\mathbf{k}} \mathbb{E}\left(B_{\mathbf{k},\varepsilon}\breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)\right).$$

Thus, it is sufficient to show the boundedness of $\mathbb{E}B_{\mathbf{k},\varepsilon}\breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)$, for any $1 \leq k \leq n$ and $\mathbf{k} = (k_0, \ldots, k_4) \in \mathcal{K}_{n-k+1}$. Indeed, noting that (2.4) implies that

$$\tilde{M} := \int_{[0,1]^{k_3}\times\mathbb{R}^{k_3+k_4}} e^{\sum_{j=1}^{k_3}\beta_j w_j + \sum_{i=1}^{k_4} u_i} d\pi_{\mathbf{k},\varepsilon} < \infty,$$

we have, for any $x \in \mathbb{R}$ and some constants $K_1, K_2 < \infty$,

$$\begin{aligned}
\left|B_{\mathbf{k},\varepsilon}\breve{f}_{k,z,t}(x)\right| &\leq \int_{[0,1]^{k_3}\times\mathbb{R}^{k_3+k_4}} \left|\breve{f}_{k,z,t}^{(\ell_{\mathbf{k}})}\right| \left(x + \sum_{j=1}^{k_3}\beta_j w_j + \sum_{i=1}^{k_4} u_i\right) d\pi_{\mathbf{k},\varepsilon} \\
&\leq \int_{[0,1]^{k_3}\times\mathbb{R}^{k_3+k_4}} \mathbb{E}\left|\widetilde{f}_{k,z}^{(\ell_{\mathbf{k}})}\right| \left(U_t + x + \sum_{j=1}^{k_3}\beta_j w_j + \sum_{i=1}^{k_4} u_i\right) d\pi_{\mathbf{k},\varepsilon} \\
&\leq \tilde{M}\lambda_\varepsilon^{-k}\mathbb{E}\, e^{U_t} \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}} e^{z+x+\sum_{\ell=1}^k u_\ell} \bar{s}_\varepsilon(u_1)du_1 \prod_{\ell=2}^k \bar{s}_\varepsilon(u_\ell)du_\ell \\
&\quad + M_{i,\varepsilon}\lambda_\varepsilon^{-k} \sum_{j=0}^{\ell_{\mathbf{k}}-2} \int_{\mathbb{R}^{k-1}} \prod_{\ell=2}^k \bar{s}_\varepsilon(u_\ell)du_\ell \max_{0\leq j\leq i} \gamma_{j,\varepsilon/2} \\
&= M_1 e^x + M_2 < \infty,
\end{aligned}$$

where the third inequality follows from (C-3). It follows that $\mathbb{E}B_{\mathbf{k},\varepsilon}\breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)$ is $O_{\varepsilon,z_0}(1)$, and so is $\mathbb{E}L_\varepsilon^{n-k+1}\breve{f}_{k,z,t}(X_{\alpha t}^\varepsilon)$. Therefore, the last integral in (C-4) is indeed $O_{\varepsilon,z_0}(t^{n-k+1})$.

(4) Plugging (C-4) into (3.1) and (3.2), and rearranging terms lead to

$$\begin{aligned}
\mathbb{E}f_z(Z_t) &= e^{-\lambda_\varepsilon t} \sum_{k=1}^n \frac{(\lambda_\varepsilon t)^k}{k!} \breve{f}_{k,z,t}(X_t^\varepsilon) + O_{\varepsilon,z_0}(t^{n+1}) \\
&= e^{-\lambda_\varepsilon t} \sum_{j=1}^n \frac{t^j}{j!} \sum_{k=1}^j \binom{j}{k} \lambda_\varepsilon^k L_\varepsilon^{j-k} \breve{f}_{k,z,t}(0) + O_{\varepsilon,z_0}(t^{n+1}). \qquad \text{(C-5)}
\end{aligned}$$

It remains to expand the coefficients

$$L_\varepsilon^{j-k}\breve{f}_{k,z,t}(0) = L_\varepsilon^{j-k}\left[\mathbb{E}\widetilde{f}_{k,z}(U_t + \cdot)\right](0) = \lambda_\varepsilon^{-k} L_\varepsilon^{j-k}\left[\mathbb{E}\widehat{f}_{k,z}(U_t+\cdot)\right](0). \qquad \text{(C-6)}$$

Using the expansion (3.7) and Remark 3.6, we have

$$\begin{aligned}\mathbb{E}\widehat{f}_{k,z}(U_t+x) &= \sum_{i=0}^{n-j}\frac{t^i}{i!}\mathcal{L}^i\widehat{f}_{k,z}(x)+\frac{t^{n-j+1}}{(n-j+1)!}\int_0^1(1-\alpha)^{n-j}\mathbb{E}\Big(\mathcal{L}^{n-j+1}\widehat{f}_{k,z}(U_{\alpha t}+x)\Big)d\alpha\\ &= \sum_{i=0}^{n-j}\frac{t^i}{i!}\sum_{l=0}^{i}B_l^i(y_0)\mathcal{L}_1^l\widehat{f}_{k,z}(x) \qquad \text{(C-7)}\\ &\quad+\frac{t^{n-j+1}}{(n-j+1)!}\int_0^1(1-\alpha)^{n-j}\mathbb{E}\left(\mathcal{L}^{n-j+1}\widehat{f}_{k,z}(U_{\alpha t}+x)\right)d\alpha.\end{aligned}$$

Finally, by combining (C-5), (C-6) and (C-7), it follows that

$$\begin{aligned}\mathbb{E}f_z(Z_t) = e^{-\lambda_\varepsilon t}\sum_{j=1}^{n}\frac{t^j}{j!}\sum_{k=1}^{j}\binom{j}{k}&\left[\sum_{i=0}^{n-j}\frac{t^i}{i!}L_\varepsilon^{j-k}\left(\sum_{l=0}^{i}B_l^i(y_0)\mathcal{L}_1^l\widehat{f}_{k,z}\right)(0)\right.\\ &\left.+\frac{t^{n-j+1}}{(n-j+1)!}\int_0^1(1-\alpha)^{n-j}\mathbb{E}\Big\{L_\varepsilon^{j-k}\big[\mathcal{L}^{n-j+1}\widehat{f}_{k,z}(U_{\alpha t}+\cdot)\big](0)\Big\}d\alpha\right]+O_{\varepsilon,z_0}(t^{n+1}). \qquad \text{(C-8)}\end{aligned}$$

Finally, since the integral in (C-8) is $O_{\varepsilon,z_0}(1)$, as seen from the uniform boundedness condition (3.18) and the estimate (C-3), we obtain that

$$\begin{aligned}\mathbb{E}f_z(Z_t) &= e^{-\lambda_\varepsilon t}\sum_{j=1}^{n}\frac{t^j}{j!}\sum_{k=1}^{j}\binom{j}{k}\sum_{i=0}^{n-j}\frac{t^i}{i!}L_\varepsilon^{j-k}\left(\sum_{l=0}^{i}B_l^i(y_0)\mathcal{L}_1^l\widehat{f}_{k,z}\right)(0)+O_{\varepsilon,z_0}(t^{n+1})\\ &= e^{-\lambda_\varepsilon t}\sum_{j=1}^{n}\frac{t^j}{j!}\sum_{p+q+r=j}\binom{j}{p,q,r}L_\varepsilon^{q}\left(\sum_{m=0}^{r}B_m^r(y_0)\mathcal{L}_1^m\widehat{f}_{p,z}\right)(0)+O_{\varepsilon,z_0}(t^{n+1}).\end{aligned}$$

□

Appendix D. Proof of Theorem 6.1

We only consider the case $x>0$ (the case $x<0$ can be done similarly by considering $\mathbb{P}(X_t\le x)$). Again, we start with the expression

$$\mathbb{P}(X_t\ge x)=\underbrace{e^{-\lambda_\varepsilon t}\mathbb{P}\left(X_t^\varepsilon\ge x\right)}_{B_t(x)}+\underbrace{e^{-\lambda_\varepsilon t}\sum_{k=n+1}^{\infty}\frac{(\lambda_\varepsilon t)^k}{k!}\mathbb{P}\left(X_t^\varepsilon+\sum_{i=1}^{k}\xi_i\ge x\right)}_{C_t(x)} \qquad \text{(D-1)}$$

$$+\underbrace{e^{-\lambda_\varepsilon t}\sum_{k=1}^{n}\frac{(\lambda_\varepsilon t)^k}{k!}\mathbb{P}\left(X_t^\varepsilon+\sum_{i=1}^{k}\xi_i\ge x\right)}_{D_t(x)}. \qquad \text{(D-2)}$$

Let f_t^ε denote the density of X_t^ε, whose existence follows from (6.4). Given that

$$\frac{d}{dx}\mathbb{P}\left(X_t^\varepsilon+\sum_{i=1}^{k}\xi_i\ge x\right)=-\frac{1}{\lambda_\varepsilon^k}f_t^\varepsilon*\bar{s}_\varepsilon^{*k}(x),$$

and that

$$\sup_x|f_t^\varepsilon*\bar{s}_\varepsilon^{*k}(x)|\le\sup_x|\bar{s}_\varepsilon^{*k}(x)|\le\gamma_{\varepsilon/2,0}\lambda_\varepsilon^{k-1},$$

one can interchange derivative and summation in (D-1) to show that for each $t \geq 0$, C_t admits a density c_t, with moreover,
(D-3)

$$\sup_x |c_t(x)| = \sup_x e^{-\lambda_\varepsilon t} \sum_{k=n+1}^{\infty} \frac{t^k}{k!} f_t^\varepsilon * \bar{s}_\varepsilon^{*k}(x) \leq e^{-\lambda_\varepsilon t} \frac{\gamma_{\varepsilon/2,0}}{\lambda_\varepsilon} \sum_{k=n+1}^{\infty} \frac{(\lambda_\varepsilon t)^k}{k!} \leq \lambda_\varepsilon^n \gamma_{\varepsilon/2,0} t^{n+1}.$$

Also, in view of Proposition III.2 in [28], there exists a real $\varepsilon_0(\eta, n) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $t \leq 1$,

$$\sup_{|x| \geq \eta} f_t^\varepsilon(x) \leq M(\eta, \varepsilon) t^{n+1}, \tag{D-4}$$

where $M(\eta, \varepsilon)$ is some constant depending only on η and ε. The last step of the proof is to deal with the terms in D_t. Recall that

$$\mathbb{P}\left(X_t^\varepsilon + \sum_{i=1}^k \xi_i \geq x\right) = \mathbb{E}\widetilde{f}_{k,x}\left(X_t^\varepsilon\right),$$

and

$$\frac{d^{(i)}}{dz^i} \widetilde{f}_{k,x}(y) = \lambda_\varepsilon^{-k} (-1)^{i-1} \bar{s}_\varepsilon^{*(k-1)} * \bar{s}_\varepsilon^{(i-1)}(x - y),$$

with

$$\widetilde{f}_{k,x}(y) := \mathbb{P}\left(y + \sum_{\ell=1}^k \xi_i \geq x\right) = \lambda_\varepsilon^{-k} \int_{x-y}^{\infty} \bar{s}_\varepsilon^{*k}(u) du.$$

Then, applying the iterated formula (2.6), we get
(D-5)

$$\mathbb{E}\widetilde{f}_{k,x}(X_t^\varepsilon) = \sum_{i=0}^{n-k} \frac{t^i}{i!} L_\varepsilon^i \widetilde{f}_{k,x}(0) + \frac{t^{n+1-k}}{(n-k)!} \int_0^1 (1-\alpha)^{n-k} \mathbb{E}\left(L_\varepsilon^{n+1-k} \widetilde{f}_{k,x}(X_{\alpha t}^\varepsilon)\right) d\alpha.$$

Using the representation of L_ε in Lemma 2.2, one can easily verify that

$$\frac{d}{dx} L_\varepsilon^i \widetilde{f}_{k,x}(y) = -L_\varepsilon^i \widetilde{f}'_{k,x}(y) = -(\lambda_\varepsilon)^{-k} L_\varepsilon^i \hat{s}_{k,x}(y), \tag{D-6}$$

$$\sup_{x,z} \left| \frac{d}{dx} L_\varepsilon^{n+1-k} \widetilde{f}_{k,x}(y) \right| \leq M_{n,k,\varepsilon} \max_{0 \leq k \leq 2n} \{\gamma_{\varepsilon/2,k}\}, \tag{D-7}$$

for some constants $M_{n,k,\varepsilon} < \infty$. Hence, one can pass d/dx through the integral and the expectation in the last term of (D-5) to get

$$\frac{d}{dx} \mathbb{E}\widetilde{f}_{k,x}(X_t^\varepsilon) = -(\lambda_\varepsilon)^{-k} \sum_{i=0}^{n-k} \frac{t^i}{i!} L_\varepsilon^i \hat{s}_{k,x}(0) + t^{n+1-k} O_{\varepsilon,k,n}(1), \tag{D-8}$$

where $O_{\varepsilon,k,n}(1)$ indicates that $\sup_x |O_{\varepsilon,k,n}(1)|$ is bounded by a constant depending only on ε, k, and n. Differentiating $\mathbb{P}(X_t \geq x)$ in (D-1) and plugging in (D-3), (D-4), (D-8) gives for any $0 < \varepsilon < \varepsilon_0$ and $t \leq 1$,

$$f_t(x) = e^{-\lambda_\varepsilon t} \sum_{k=1}^n \frac{t^k}{k!} \sum_{i=0}^{n-k} \frac{t^i}{i!} L_\varepsilon^i \hat{s}_{k,x}(0) + t^{n+1} O_{\varepsilon,\eta}(1),$$

where $O_{\varepsilon,\eta}(1)$ is such that $\sup_{t\leq 1}\sup_{|x|\geq\eta}|O_{\varepsilon,\eta}(1)|<\infty$. Rearranging the terms above leads to

$$f_t(x)=e^{-\lambda_\varepsilon t}\sum_{p=1}^{n}c_p(x)\frac{t^p}{p!}+t^{n+1}O_{\varepsilon,\eta}(1),$$

with

$$c_p(x):=\sum_{k=1}^{p}\binom{p}{k}L_\varepsilon^{p-k}\hat{s}_{k,x}(0).$$

The expression in (6.10) follows from the Taylor expansion of $e^{-\lambda_\varepsilon t}$, using also that $\sup_x|c_p(x)|<\infty$ (a fact which follows from (D-6)). Finally, the "constant property" of (6.10), for any $0<\varepsilon<\varepsilon_0$, follows from inversion. Indeed, given that a posterior

$$f_t(x)=\sum_{k=1}^{n}\frac{a_k(x)}{k!}t^k+t^{n+1}O_{\eta,\varepsilon}(1), \tag{D-9}$$

holds true for any $t\leq 1$ and $0<\varepsilon<\varepsilon_0$, $a_k(x)$ can be recovered from $f_t(x)$ (independently of ε) by the recursive formulas:

$$a_1(x)=\lim_{t\to 0}\frac{1}{t}f_t(x),\quad a_k(x)=\lim_{t\to 0}\frac{k!}{t^k}\left(f_t(x)-\sum_{i=1}^{k}\frac{a_i(x)}{i!}t^i\right),\quad 2\leq k\leq n.$$

□

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