STATISTICS AND MATHEMATICS: TENSION AND COOPERATION

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1. INTRODUCTION. It has become a truism, at least among statisticians, that while statistics is a mathematical science, it is not a subfield of mathematics. We even have aphorisms to express some ways in which our science differs from mathematics. George Box: "All models are wrong, but some are useful." George Cobb: "In mathematics, context obscures structure. In data analysis, context provides meaning." David Moore: "Mathematical theorems are true; statistical methods are sometimes effective when used with skill." That versions of these aphorisms apply whenever mathematics models phenomena in another field only emphasizes that statistics is another field. Cobb and Moore [**2**] discuss the implications of this fact for the teaching of statistics.

Our present focus is neither the scientific distinction between mathematics and statistics nor the teaching of these subjects. We are concerned with the environment in which the fields now operate and the implications of our environment for our future prospects and for opportunities for synergy. We believe in particular that increased cooperation between the American Statistical Association (ASA) and the MAA could serve the interests of both mathematicians and statisticians.

Both mathematics and statistics are of course served by other major societies, the American Mathematical Society (AMS) and the Institute of Mathematical Statistics (IMS). These are in fact the most important societies in supporting academic research, and as such they play a large role in the culture of the two fields. The AMS is particularly significant, if only because most mathematicians are academics and most statisticians are not. Our recommendations focus on the ASA and the MAA because these groups have been more active in areas such as undergraduate education and because more statisticians are involved in the MAA than in the AMS.

We begin with a simple thesis: statistics has cultural strengths that might greatly assist mathematics, while mathematics has organizational strengths that can provide shelter for academic statistics, shelter that may be essential for its survival. Better relations between these two connected fields could help both. It is mainly cultural differences that prevent closer relations. We might say that mathematics is French, while statistics is resolutely Anglo-Saxon. The French, proud of a long history and high culture, are wary of aggressive Anglo-Saxon pragmatism. In the words attributed to a French diplomat, "It works in practice, but does it work in theory?" Cultural change is never easy, but there are signs that mathematicians are recognizing the need for some change. In doing so, they may (inadvertently?) rescue statistics from oblivion.

In developing our thesis, we serve up some potentially unpalatable implications, both for statisticians and for mathematicians. Academic statisticians will have to give up dreams of autonomy and seek shelter within the organizational strengths of mathematics. Worse yet, just at a time when various developments have made the practice of statistics less dependent on mathematics than in the past, it may be time to recognize that mathematics offers statistics intellectual as well as institutional protection.

Thus the statisticians who approach our table are to be served humble pie. What do we offer our mathematician colleagues? We urge them to see their culture as others see it, and change. Worse yet, we offer our own field, statistics, as a model for that change. How do others see mathematics? The AMS recently spent seven years interviewing the chairs of research mathematics departments and their deans. Here [**6**, p. 65] is what the deans said:

> The prevalent theme in every discussion was the insularity of mathematics. Mathematicians do not interact with other departments or with faculty outside mathematics, many deans claimed, and they view this as a problem both for research and for teaching. In many cases, deans contrasted mathematics with statistics, which they pointed out had connections everywhere.

In suggesting that mathematics has become insular and statistics imperiled, we invite debate, but we attempt to argue from data and hope others will do likewise.

2. STATISTICS IS DIFFERENT. We begin by outlining the ways in which statistics is the healthier discipline. Among the encouraging vital signs, we find increasing enrollment and a consensus on teaching, more non-academic employment and links to many academic fields, and a positive response to technological change.

Increasing enrollment. The 1995 CBMS survey [**19**] details the enrollment situation: even if we look only at mathematics and statistics departments, the number of students taking elementary statistics is growing rapidly while, roughly speaking, everything else mathematics departments do is eroding.

Elementary Statistics Enrollment (thousands)

	Math depts	Stat depts	2-year colleges	Total
1990	87	30	54	171
1995	115	49	72	236

According to the same survey [**19**, p. 90] the ratio of statistics enrollment to calculus I enrollment in two-year colleges (where mathematics department enrollment is increasing) rose from 56% in 1990 to 82% in 1995. In the last three years, the birth and rapid growth of Advanced Placement statistics—roughly 7,500 exams in 1997, 15,486 in 1998, 25,240 in 1999, and 38,000 expected in 2000—means that for the first time large numbers of academically ambitious secondary school students are being exposed to modern data-centered statistics. Some will decide to study more of it.

Meanwhile, the numbers of mathematics majors and upper division enrollment have eroded seriously [**9**]:

Mathematics Enrollment (thousands)

	1985	1990	1995	% Change 1990–1995	% Change 1985–1995
Calculus	637	647	539	−17	−15
Advanced	138	119	96	−19	−30

The upward trend in statistics enrollment takes place in the context of broad agreement about what should be taught, especially in the beginning course.

Consensus on teaching. Although it would be rash to claim that any group of academics has reached a consensus on any subject, statisticians have by academic standards a clear consensus on the important subject of what we should teach beginners. As Richard Scheaffer says [**27**, p. 156], "With regard to the content of an introductory statistics course, statisticians are in closer agreement today than at any previous time in my career." Contrast the wars over "reform calculus" among mathematicians.

Doctoral programs in statistics have occasioned more debate, but have evolved toward a general pattern of three equal tracks of core material: probability, statistical theory, and methodology. Equal treatment for methodology (generally both classical and computer-intensive) is relatively recent. Contrast the fact that the AMS Committee on the Profession has been unable to issue a report on the question "should the AMS advocate for more interdisciplinary research and correspondingly broadened graduate programs, and if so, how?" because of disagreement among its members [**17**].

Even in responding to the invitation of the National Council of Teachers of Mathematics to comment on the draft revision of its important *Standards* [**22**], the lack of consensus among mathematicians can be seen by contrasting the MAA group's statement that "we worked hard, and successfully, to obtain consensus on our reports" [**26**] with the AMS response that it "found that we are far from being of one mind on many issues" [**12**]. The ASA group quickly agreed on united reports.

Why is statistics enrollment increasing? Why has a consensus on teaching emerged? Causes are no doubt multiple, but both trends are surely linked to a growing recognition that statistics is a subject whose goal is to solve real-world problems. This sense of the subject is also reflected in its links to other academic subjects, and in the opportunities for non-academic employment for those who study statistics.

More non-academic employment. The 1999 AMS-ASA-IMS-MAA survey [**18**] shows continuing differences between the employment of new Ph.D.s in core mathematics and in statistics. We thank James Maxwell of AMS for providing an advance copy of this report. For 1998–1999 doctoral recipients employed in the United States:

Nonacademic employment of new Ph.D.s by field of thesis

Statistics	PAN	All others
$\frac{83}{167} = 49.7\%$	$\frac{64}{157} = 40.8\%$	$\frac{79}{435} = 18.2\%$

Here PAN aggregates the fields of probability, applied mathematics, and numerical analysis. "All others" may be taken as core mathematics. The distinctions among fields in non-academic employment, though still marked, have in fact somewhat diminished in recent years as more core mathematics Ph.D.s have sought non-academic employment. There is also considerable year-to-year variation: the percentages of non-academic employment in the three groups were 54.7%, 46.5%, and 29.4% for 1997–1998 doctoral recipients. However, this variation does not change the relative standings.

Data from the same survey show that the percentage of Ph.D.s going to women remains higher in statistics than in mathematics:

Female Ph.D.s in the mathematical sciences

Statistics	All other fields
35.7%	25.7%

Though data are lacking, we conjecture that the difference is related to the greater opportunities for non-academic employment in statistics.

The survey data on employment understate the contrast between fields because they pertain only to recipients of the Ph.D. Statistics has a meaningful and very employable professional masters degree designed to train statisticians for non-academic work. Most statistics departments distinguish the professional M.S. program from the first portion of the Ph.D. program (which may also lead to an M.S. degree). Although data are hard to come by because working statisticians with an M.S. degree do not commonly join professional societies, it appears that in terms of counts of active professionals, mathematics and statistics are disciplines of similar size. Statistics is a much smaller field in academe, but mathematics is the smaller field once we leave campus.

Statisticians' greater opportunities for non-academic employment have parallels on campus, where statisticians collaborate with colleagues in many other areas.

Links to many fields. Mathematicians are justifiably attracted by the abstract beauty of their subject, which Bertrand Russell characterized as cold and austere, like the beauty of sculpture. But in the past century, and especially in the last 50 years, pursuit of abstract beauty has often meant turning away from connections to other subjects. Mathematicians have become notorious, as their deans told the AMS interviewers, for reluctance to talk with researchers in other disciplines. Even a panel of very senior mathematicians [**24**] agrees: "Communication between mathematical scientists and other scientists is poor the world over."

The panel of course fudged a bit: "mathematical scientists" other than core mathematicians often work in environments where communication is essential. This is certainly true of non-academic statisticians, that is, of most statisticians. Statistical work in the private sector is almost always interdisciplinary, centered on the needs of clinical trials, market research, or industrial process improvement rather than on the internal development of statistics itself. Academic statisticians have gotten religion more slowly, but many now display the enthusiasm of converts. Statistics departments have long operated consulting services that both serve their campuses and train graduate students. As the culture of academic statistics has changed, leading-edge research looks less like research in mathematics and increasingly interdisciplinary and methodological, often driven by problems from other fields. It is common to find highly regarded young statistics faculty doing research in genomics, computational finance, neuroscience, data compression, and so on. Roughly half the faculty in the top-rated Stanford department hold joint appointments in other disciplines. At Carnegie Mellon, all statistics Ph.D. students are required to complete a semester-long interdisciplinary project in which they collaborate with someone outside their own department. The National Research Council's report *Modern Interdisciplinary University Statistics Education* [**23**] grew out of a symposium held almost a decade ago. In the last dozen years the NRC's Committee on Applied and Theoretical Statistics has held symposia and published proceedings on a variety of emerging areas of research where statistics meets one client field or another.

Although statistics has been more outward-looking than mathematics throughout the past century—the only century of its existence as a separate field—its current state of energetic outreach and fruitful linkage to other subjects is a comparatively recent development, one that owes much to computers.

Energized and redirected by computing. The change in the culture and content of academic statistics has occurred in large part because of technology, which has revolutionized the teaching of statistics, reshaped the practice of statistics, and created new types of research questions.

Before computers, teachers of statistics had essentially two choices—derivations or algorithms. Some courses emphasized proofs, and lost all but the most mathematically inclined; others emphasized arithmetic, and bored all but the most desperate to pass. Neither emphasis did much to show statistics as practically useful or intellectually interesting. Perhaps it is no wonder that consensus emerged comparatively quickly once computers made it possible for students to work with real data.

The practice of statistics also felt the profound effect of computing. Now that you could do one analysis quickly, it was possible to analyze a data set in more than one way and compare the results. Once you had entered the data into a computer, the marginal effort required for additional analyses was minimal. Multiple analyses led to greater emphasis on assumption-checking and model-fitting, which in turn opened up research areas in diagnostics and statistical graphics.

Academic statistics responds (eventually) to changes in professional practice because of the importance of non-academic employment in the field and the tradition of offering consulting on campus. But the primary driver of change has been the intellectual challenge of new types of problems.

Academic statistics in the 1950s and 1960s looked inward, drawn by hopes that inductive inference could be satisfactorily mathematized. The outward pull of statistical practice is now much more noticeable. One example: 40 years ago Bayesians were working to show that their approach had desirable properties not shared by non-Bayesian methods. Although the arguments were mathematically rigorous, they converted few practitioners. Bayesian posterior distributions were simply too hard to compute in applied settings of even modest complexity. High-speed computing made it possible to revisit the computational challenge in a new environment, and academic research responded. In the last decade, Markov Chain Monte Carlo and other computer-intensive methods have made the computation of Bayesian posterior distributions much more nearly automatic. Bayesian methods are now increasingly used in practice, and "Bayesian computation" has become an active research specialty.

Computational resources that by the standards of the past are infinite and free, combined with scanning technologies in other fields that produce previously unthinkable volumes of data, open new problems concerning analysis of massive data sets; for example, see **[10]** and **[7]**. Clever use of computing has simultaneously changed the way small sets of data are analyzed, as when resampling methods allow large-scale computation to replace hard-to-justify model assumptions **[5]**. The result has been a dramatic shift in research emphases as technology moves statistics back towards its roots in data analysis and scientific inference. Links to other fields have strengthened, as freedom from restrictive assumptions has brought greater flexibility and utility to what statisticians can offer colleagues in the sciences. At the same time, computationally intensive methods (bootstrap, neural nets, wavelets) have come to be seen as core statistics. One apparent corollary is that applied work in statistics is less dependent than in the past on mathematically derived analytic solutions. We argue, however, that this important new freedom should not be mistaken for independence from mathematics.

3. STATISTICS IS, ALAS, DIFFERENT. Should statisticians gloat? Not at all. Each of the strengths we have noted has a darker side. The picture that emerges when all the pieces come together is one of organizational weakness. The advantages of statistics over mathematics in our current environment are cultural, and cultural strength rarely outweighs organizational weakness. God is on the side of the big battalions. Mathematics has the big battalions, and statistics has a few guerrillas scattered about the academic jungles. Mathematics is far likelier than statistics to have an extended, if not prosperous, future.

Weak organizational players. Statistics enrollment may be increasing, but it remains the case that many more statistics courses are taught in mathematics departments than in statistics departments. Introductory statistics is also taught in departments of psychology and economics and in schools of business and engineering.

Statisticians active in reform have reached a consensus on teaching, but they lack institutional power. Neither of their major professional organizations matches the MAA's emphasis on teaching. The IMS primarily supports academic researchers, while the ASA has a majority of non-academic members and an emphasis on serving working statisticians. The ASA's Section on Statistical Education is one of 21 sections, and only about 1,000 of ASA's roughly 17,000 members belong to it.

Statistics does have links to many fields, but this is because it is a methodological discipline rather than a core substantive area. As a result, statistics in most colleges is distributed among many core disciplinary departments (including mathematics), and is taught and practiced by faculty whose main training and interests lie elsewhere. Most large research universities have separate statistics departments, but these are typically small relative to other sciences and are rarely viewed as essential in the way that a mathematics department is essential. Statistics departments are natural targets when budget crunches demand downsizing.

This is also true outside academe. Although there are many more non-academic than academic statisticians, many corporations (Dupont, Lipton, Corning, Kodak, ...) have reduced or eliminated separate statistics units. Technology enables engineers and other directly productive employees to do much more statistics than in the past. In a survey of the state of statistics in business and industry, Gerry Hahn and Roger Hoerl of General Electric speak of "statistics without statisticians" [**11**]. Statistics is increasingly universal, but "simply performing a statistical analysis is no longer a marketable task—anyone with a laptop can do that." The situation is not entirely bleak, as discussants to Hahn and Hoerl's paper note, but the strong employment situation for statisticians is disproportionately concentrated in the pharmaceutical industry and should not be taken for granted.

The organizational weakness of statistics, growing as it does out of the methodological nature of the subject and its consequent close ties to many fields, raises a troubling question.

Does the field have a core? The parochialism of mathematicians may be unwise, but it is explained and in part justified by the long history and continuing triumphs of a deep discipline. Mathematicians, and even some others, understand that this inward-looking field produces both profound beauty and "unreasonably effective" tools for other sciences. Statistics, in contrast, coalesced in this century from beginnings in many fields and may be about to dissipate back into many fields. It isn't clear what "statisticians" engaged in market research and molecular biology have in common. To the extent that applied problems—that is to say, research topics—in one client discipline become specialized and thus different in substance from problems in other client disciplines, statisticians become more identified with the area of their applied work than with statistics as a free-standing discipline. A pessimistic (or perhaps simply realistic) vision of the future sees statistics dissipating back into other fields.

Undergraduate mathematics has experienced a similar, though milder, dispersal. Garfunkel and Young [**8**] documented the migration of the teaching of mathematics beyond calculus to other academic departments. That their work has gone largely unremarked, while the CBMS survey shows continuing attrition in advanced mathematics enrollment, is a disturbing cultural indicator. The dearth of students concentrating in mathematics, which has not gone unremarked, is another indicator of poor health. If academic statistics departments risk extinction, mathematics risks a future in which it looks to others much like philosophy: an old and respected discipline engaged in intense investigation of questions of interest only to itself. Mathematics departments will no doubt continue to have larger service teaching loads than philosophy departments, cold comfort though this may be.

The threat from information technology. Although statistics has been re-energized and redirected by computing, it now risks being engulfed by information technology. Friedman [**7**] and others have noted that new areas in which statistical ideas offer promise are being pursued more vigorously by non-statisticians than by statisticians. Information technology is now the most important methodology for most scientific fields, displacing both statistics and mathematics from their traditional roles. Mathematics provides methodology on the side, so to speak, and will therefore survive. Statistics is inherently a methodological discipline, and is therefore at risk.

4. A BASIS FOR SYNERGY: IN THEORY, IT CAN WORK. Why shouldn't statistics simply allow itself to be reabsorbed by its client fields, or, alternatively, swallowed whole by information technology? From the other direction, why should mathematicians want to retain statistics under their organizational umbrella? Given existing cultural differences, real cooperation seems unlikely unless there are shared commitments at a deep level. Fortunately, there are. In what follows, we discuss four of them.

Undergraduate mathematics courses. Statisticians depend on, and care about, the undergraduate program in mathematics. Indeed, many faculty at top graduate departments of statistics care more about whether applicants have taken a good course in real analysis than about whether they have taken a good course in statistical methods. Although the ASA is mounting an initiative to strengthen undergraduate programs in statistics, this will in no way reduce the importance of undergraduate mathematics to statisticians. Quite the contrary. A student headed for any of the better graduate programs in statistics should have taken at least seven courses in mathematics: three in calculus, plus one each in linear algebra, probability, mathematical statistics, and real analysis. More is better. (Contrast this with how little mathematics is currently required for computer science.) If undergraduate programs in statistics grow, mathematics enrollment will benefit.

Goals for undergraduate teaching and learning. The growth of statistics enrollment is part of a larger pattern, with parallel growth in computer science and economics. The combined increases in these areas more than offset the erosion in mathematics enrollment. One inward-looking response by mathematics shrugs off the pattern as part of a shift toward a vocationalism unworthy of the Liberal Arts. Another view is this: traditional mathematics exposition and teaching often presents the subject's abstractions as completed structures. Examples come after, to serve as illustrations. Students receive a finished product, but don't participate in the *process* of building it. What statistics and other quantitative subjects offer, and traditional mathematics courses often do not, is more experience with the process of searching for patterns at a low level of abstraction before formulating a more abstract statement and then assessing its validity. Mathematicians rely on this process, of course, and there is no obstacle to their making it a more explicit part of the undergraduate mathematics curriculum. At Mount Holyoke College, for example, all mathematics majors are required to take a sophomore-level course devoted to this process, entitled "Laboratory in Mathematical Experimentation" [**21**]. When students experience the process, as opposed to just its products, the barriers between learning and research are lowered in healthy ways.

This observation is of course far from original. Constructivism migrated long ago from schools of education to departments of mathematics, and a parallel reform movement has urged science teachers to make their laboratory assignments more a process of discovery—

an experience of scientific method in action—than a ritual confirmation of canonical results. The key point for our argument is that mathematicians and statisticians share a commitment to a process of pattern searching, generalization, and verification that operates at a deep level, despite surface differences.

Mathematicians who care about teaching are struggling with a variety of issues, among them the roles of technology, applied context, and the prerequisite structure that relates courses to each other [**25**]. Statisticians who care about teaching face parallel questions, but because computing is more central to what statisticians do, computers began to change approaches to teaching in statistics much earlier and more profoundly than in mathematics. For example, Minitab, a statistical software package explicitly designed to be used in teaching, was developed in the 1960s. By the early 1970s, many university departments were using computers in some of their courses and beginning to think systematically about how to bring them to the introductory level. By the mid-1980s, some introductory textbooks included computer exercises. Statisticians did not reach consensus on the introductory course without a struggle, but our struggle began much earlier than in mathematics and has been going on longer, with a greater sense of urgency. Thus statistics may be able to offer some useful models to mathematics.

The power of mathematics in statistics. Statistics, like physics and economics but unlike algebraic topology or probability theory, values mathematical understanding as a means to an end, not as an end in itself. Like physics and economics, but unlike sub-fields of mathematics, statistics has a subject matter of its own, quite apart from mathematics. While it is thus true that statistics is not a subfield of mathematics, we suggest that continued emphasis on this truth fights a battle that has been won and risks losing a more important battle. Even if statistics resists fragmentation and dispersal among its client disciplines, it still risks absorption by information technology. Ironically, a major reason for that threat is the increased intellectual distance of contemporary statistics from mathematics. Technology has greatly expanded what statistics can do, but it has also made advanced statistical tools usable by people with little knowledge of either mathematics or statistics.

Twenty years ago, selection of variables in multiple regression and several-way analysis of variance with unbalanced data were specialist topics. People who attempted such analyses were likely to have taken several statistics courses. Then, as now, statisticians insist that they have insights that go beyond suggesting appropriate methodology and that some grasp of these insights is required for skillful use of statistical software. However, today's software is much easier to use, and many introductory courses give students extensive practice. Whereas once there was a good chance that a person using multiple regression software would know a fair amount of statistics and undergraduate mathematics, that is no longer the case.

We do not deny that accessibility and wide use of statistical methods are good for science and society as a whole. But there are costs associated with this good. Elaborate studies using complex statistics, when done without much background in mathematical or statistical thinking, are prone to errors that lie deeply buried in the awful details. This is of course true whenever complex mathematical models are automated and then used without adequate grounding in mathematics and in the substance that the mathematics describes. It seems clear that *in the debate over the relative roles of thought and automated methods, statisticians* *and mathematicians are natural allies.*

Statistics is distinguished from "mere computing" by its extensive use of mathematical models. Classical statistical inference is largely based on the "general linear model" combining n-dimensional Euclidean geometry for structure with Gaussian distributions to model variation. The recent history of statistics is studded with examples where existing mathematical structures provide a powerful and elegant new understanding, or where the process of seeking a natural level of generality unifies a diverse collection of examples whose connections had not previously been seen clearly. The use of differential geometry to understand exponential families of distributions is a good example of the first kind (e.g., [**16**]); the emergence of the EM algorithm is an instance of the second [**3**].

We have earlier [**2**] given examples demonstrating that mathematical knowledge is insufficient for statistical understanding. We now offer a more advanced example that illustrates that mathematical knowledge is often necessary.

An example. The following example illustrates first, the shortcomings of "old-style" statistical methods, which require simplifying assumptions to make a problem analytically tractable; second, a computer intensive alternative, which requires no such simplifying assumptions, but which nevertheless makes substantial use of mathematics; third, an unexpected role for (elementary) abstract algebra; and finally, an illustration of a trap for the unwary user who neglects essential mathematics. This example is adapted from a recent article by Diaconis and Sturmfels [**4**]. The mathematics is theirs, the moral subtext is ours.

Consider testing independence of the row and column variables in a two-way table of counts. For example, to test for association between birthday and death day, let o_{ij} be the number of people in a population of interest who were born in month i and died in month j ($i, j = 1, 2, \ldots, 12$). Diaconis and Sturmfels analyze such a table for 82 descendants of Queen Victoria. Compute row and column sums o_{i+} and o_{+j} and the grand total o_{++}, and use these to find the expected counts under the assumption of independence, $e_{ij} = o_{i+}o_{+j}/o_{++}$. Now compare observed and expected, by computing the Pearson statistic $X^2 = \sum(o_{ij} - e_{ij})^2/e_{ij}$. X^2 is a measure of how far the observed values fall from what we would expect if birth and death months were independent. The p-value is the proportion p of tables, among all those with the same row and column sums, for which the value of X^2 equals or exceeds the value computed from the data. Thus the p-value measures how surprising we should find the data if we believe the hypothesis of independence. If the p-value is small, we have strong evidence against that hypothesis.

How shall we calculate the proportion p? The traditional method, taught in most introductory courses on statistical methods, is the chi-square test. Make the simplifying assumption that the cell counts are large enough to justify an asymptotic argument that replaces multinomial distributions with Gaussian approximations. Then the p-value is approximately equal to the probability that a chi-square random variable with $121 = (12 - 1) \times (12 - 1)$ degrees of freedom is greater than or equal to X^2.

This is "old-style" statistics, in that simplifying assumptions that may or may not fit the data at hand are needed to render the problem analytically tractable. In this case, a lot gets lost in the rendering. One version of the simplifying assumption often used in practice is that most expected cell counts e_{ij} are at least 5; see [**20**, section 3.2.5] for a discussion.

For our data, with an average cell count of 82/144, the assumption is not appropriate. The old-style approach, “Make simplifying assumptions until you have turned the problem into one you can solve analytically”, has sometimes appeared to put “use mathematics” ahead of “get the job done right.” Reaction against this assignment of priorities is part of what has energized declarations of independence by statisticians.

For very small data sets, you don’t need the chi-square approximation because exact enumeration is possible. “Fisher’s exact test” uses the hypergeometric distribution as a shortcut, but the method is equivalent to enumeration. Enumeration avoids the unrealistic assumption of large expected cell counts, but unfortunately even computer-aided enumeration is practical only in small problems. There are simply too many possibilities. Diaconis and Sturmfels give a 4×4 data set with 592 observations for which 1.2×10^{18} tables have the same row and column totals. Alternatively, or in addition, there may be specific constraints that make enumeration difficult. For tables in three or more dimensions, for example, the hypothesis of interest may not be as simple as independence of the individual dimensions, and the constraints may involve several overlapping two-dimensional arrays of totals to be matched. Or there may be ”structural zeros” in the data, e.g., in a table with dimensions for gender and for site of cancer. The fact that computer-aided enumeration is not often used for problems of this sort highlights the point that in statistics “computer-intensive” means something more subtle than just very-high-speed plodding.

The current computer-intensive method for computing p-values for this problem is known as Markov Chain Monte Carlo, the same method mentioned earlier for computing Bayesian posterior distributions. To apply it here, define a connected, aperiodic, reversible Markov Chain whose states are all the tables of size 12×12 that meet the required constraints (e.g., with non-negative integer entries having the given row and column totals). One standard way to do this is to assign transition probabilities as follows. Pick two rows i and j and two columns r and s at random. Move from the current data table by adding 1 to cells (i, r) and (j, s) while subtracting 1 from (i, s) and (j, r); stay in place if the chosen move would produce a negative entry. This chain has a known equilibrium distribution, which can be modified (by accepting or rejecting moves using coin flips with suitable bias) to give the desired sampling distribution for the set of tables. Run the (modified) chain long enough to reach equilibrium, then sample from it, say, every 50 steps. Approximate the p-value by the proportion of sampled tables whose X^2 value equals or exceeds the one computed from the data.

Clearly, this elegant computer-intensive method could not have been developed without mathematics. More to the point, even its use is made easier, and in some instances made possible, by mathematical understanding that is not at all obvious. A major contribution of the paper by Diaconis and Sturmfels is to show that defining the Markov Chain (finding a Markov basis) is equivalent to finding the generators of an ideal in a polynomial ring. Thus methods of computational algebra, and a computer system like Maple, can be used to set up the Markov Chain.

As a final point, we repeat an observation of Diaconis and Sturmfels, that connectedness of the chain really matters, and cannot be taken for granted. They cite a published example where inattention to this detail led to an incorrect p-value.

Applied problems as an inspiration for research. In theory, at least, mathematics and statistics share a commitment to developing new tools to solve applied problems. In statistics, attention to the needs of other fields has always been responsible for much of the profession's evolution. In the long history of mathematics as well, applied problems were traditionally the source of most new mathematics. In theory, yes; in statistics, yes; in history, yes. In practice, in mathematics, at the present time, research appears to be disproportionately driven by the internal development of mathematics itself. We hope that mathematicians will find the history of their own subject persuasive, the specter of academic philosophy galvanizing, and therefore, the example of statistics useful.

5. CAN SYNERGY WORK IN PRACTICE? In theory, as we have seen, there is a substantial basis for cooperation:

- Graduate programs in statistics need undergraduate programs in mathematics, at a time when mathematics programs need to be needed.
- Despite intellectual differences, mathematics and statistics both depend on the process of working from the concrete to the abstract, and can learn from each other's successes and failures in teaching this process to undergraduates. In addition, because statisticians were confronted much earlier by computer-driven challenges in curriculum and pedagogy, mathematicians may be able to benefit from their experience.
- Statistics can benefit from embracing more openly the importance of mathematical thinking, particularly in research and practice near its boundary with computer science, where statistics is especially threatened.
- At the same time, mathematics can benefit from the experience of statistics over the last 30 years, during which computing has redirected energies back towards the roots of statistics in applied science. A parallel shift in orientation for mathematics would be consistent with its nineteenth-century history of ties to the sciences.

Can cooperation work in practice? The recent experience of mathematics at the University of Rochester ([**13**], [**14**]**,** [**15**]) offers some encouragement. The organizational strength of mathematics, via the Rochester Task force of the AMS, played an influential role in rescuing the department from a threatened elimination of the Ph.D. program and 50% cut in faculty. The key to the compromise "Renaissance Plan" for the mathematics department emerged from what the university's president described [**13**] as "a series of unprecedented conversations between math faculty and the administration and between math faculty and their colleagues in other departments. That, in turn, led to the Department of Physics and Astronomy's offer to promote linkages by joint appointments and a new dedication on the part of the mathematics faculty to strengthen undergraduate instruction and their ties to other departments . . . ."

In the remainder of this section, we offer two clusters of remarks on cooperation between mathematics and statistics, first recounting some good beginnings, then suggesting some promising continuations.

Good beginnings. Cooperation between professional societies can help bridge the culture gap and help mathematics and statistics to profit from their complementary strengths. Substantial cooperation between ASA and MAA is already in evidence.

Joint Committee. In 1992 the MAA and ASA formed a joint committee on undergraduate statistics. This group has done useful things, and statisticians within the MAA have done more: workshops for mathematicians teaching statistics, volumes on statistics published by the MAA, lobbying for adequate treatment of statistics in MAA guidelines for departments. The MAA has been generally receptive to the concerns of its statistician members.

SIGMAAs. The ASA has an active "isolated statisticians group" that supports statisticians teaching in mathematics departments, where they lack statistical colleagues. Activities include meetings at the Joint Statistical Meetings and elsewhere and an active listserve (isostat@oberlin.edu). The MAA is now forming SIGMAAs, special interest groups analogous to ASA's sections. A statistics education SIGMAA will be among the first formed.

Guidelines. The MAA *Guidelines for Programs and Departments in Undergraduate Mathematical Sciences* can greatly influence the reception offered statistics and statisticians in mathematics departments. The current draft offers a warm reception. For example, it urges recognition of consulting as scholarly work and stresses that instruction in statistics and other border areas should be developed and supervised by faculty fully trained in the area. Acceptance of these recommendations by the MAA would be an important step in changing culture.

Annual meetings. The ASA/MAA joint committee has contributed to a greater presence of statistics at the Joint Mathematics Meetings. The MAA has been generally hospitable to statistical sessions and short courses, and in each of the last few years there has been both a well-attended minicourse related to the teaching of statistics and a well-attended contributed paper session with enough papers to fill six hours spread over three days.

Suggestions for the future. Here are some suggestions for the MAA, for the AMS and the MAA jointly representing the mathematics profession, or for mathematical and statistical societies working together, to build from our promising beginnings.

Teach statistics well. Consider these data for comparable departments on the percentage of elementary statistics enrollments treated to two "good practices" [**19**, p. 71]:

Elementary Statistics (1995)

	Ph.D. Math Depts	Ph.D. Stat Depts
Tenure-track instructor	29%	46%
Computer assignments	42%	61%

The performance of the statistics departments is nothing to be proud of, but the gap between mathematics and statistics departments is clear. Statistics is almost the only offering of mathematics departments that is growing. Self-interest alone suggests that it be given greater attention.

The ASA/MAA joint committee has taken the lead here; see its report [**1**]. The ASA Board of Directors has unanimously endorsed a short version as guidance for beginning instruction in statistics. We hope that the MAA will look closely at this issue as well. Because more statistics teaching is done by mathematics departments than by statistics departments, statisticians have a major stake in how mathematics departments teach statistics. Mathematics departments should want to offer modern data-centered elementary statistics courses to satisfy what is now their fastest-growing clientele.

Undergraduate programs. If statistics *courses* are attracting more student interest and the rapid rise of AP Statistics promises more interest, development of undergraduate statistics *curricula* is the next step. In all but large institutions, where full undergraduate major programs in statistics are possible, the focus may be on interdisciplinary concentrations in statistics that draw on quantitative elements from several disciplines. Mathematics departments can take the lead on their campuses, and ASA and MAA might cooperate to provide thoughtful guidelines.

Consulting and joint appointments. Statisticians have learned how to organize and fund successful consulting services, and joint appointments are common in statistics departments. Consulting services are a good indicator of how well a statistics department connects with its campus, as they not only serve many disciplines but also lead almost inevitably to joint research. A campus that, like most, lacks a separate statistics department offers an opportunity to the mathematics department: start a consulting service. Statisticians on the mathematics faculty may be the first to draw business, but many areas in mathematics are relevant to the work of scholars in other fields. Of course, the department must be willing to recognize consulting on and off campus as scholarly work, a change of attitude for many mathematicians.

Research and graduate programs. Statistics graduate programs are vastly different from those of the 1960s, with much more emphasis on computational, methodological, and applied research. Most statisticians understand that "applied research" is applied *to* something; this is not traditional usage in mathematical circles. The resistance of many mathematicians to "more interdisciplinary research and correspondingly broadened graduate programs" (to borrow the phrasing of the AMS Committee on the Profession) paves the road to becoming another philosophy department. Due to the differing nature of the disciplines, the culture of the relatively few leading research departments in mathematics and statistics may well diverge. Leading statistics departments already have strong programs of methodological and interdisciplinary research. Their counterparts in mathematics, most of which are at institutions with separate statistics departments, tend to concentrate on core mathematics. Other mathematics departments, especially those in institutions without separate statistics departments, may find it attractive to offer graduate concentrations in statistics, operations research, and applied mathematics. Their graduate programs will naturally then evolve in directions that no longer follow the path of leading departments. Continued AMS discussion aimed at legitimizing graduate programs that do not attempt to imitate the few leading research institutions would be welcome.

Publications. It may be that individual mathematics and statistics societies lack the scale needed to succeed in the new environment of on-line publication, which favors groups that can offer libraries searchable archives of large numbers of journals and other publications. Exploration of possibilities for cooperation among societies is natural.

Annual meetings. A good start has been made, as noted earlier, but much remains to be done. The January Joint Mathematics Meetings are important both for statisticians seeking academic employment and for mathematics departments seeking to hire statisticians, as the meetings are the natural locus for job interviews. The Joint Statistical Meetings, held in August, cannot fill this role. The timing of the two major meetings illustrates once more that mathematics is primarily an academic discipline and that statistics is not. ASA can contribute here by inaction: it should encourage attendance at the mathematics meetings rather than, as has been suggested, sponsor a winter statistics meeting. AMS and MAA can lend force to ASA's inaction by doing still more to fill the vacuum left for statisticians by the absence of ASA meetings in January.

A final plea. Having made several pleas to mathematicians, we end with a simple plea to academic statisticians: **Join the MAA**. To succeed in and change a foreign culture, immigrants must be willing, at least partially, to assimilate.

6. CONCLUSION. Mathematics, a core discipline, looks inward and risks being seen as increasingly irrelevant. Statistics, a methodological discipline, looks outward but risks being swallowed by information technology. Both professions have a stake in the survival of statistics as a subject informed and structured by mathematics. To mathematics, statistics offers not only the example of an outward looking culture, but also entree to new problems ripe for mathematical study. To statistics, mathematics offers not only the safe harbor of organizational strength, but intellectual anchorage as well: mathematical understanding is an essential part of what distinguishes statistical thinking from most of the rest of information technology. Increased cooperation between mathematical and statistical professional associations can lead the societies, their members, and their disciplines in healthier directions.

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