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Stochastic lead time with order crossover

Ching-Chi Yang[a] and Dennis K. J. Lin[a]

[a]The Pennsylvania State University, University Park, PA, USA

ABSTRACT

Lead time plays an important role in many areas, including supply chain, economics, and marketing. A conventional assumption in most stochastic lead-time inventory models is that the lead times are independent and identically distributed (i.i.d.). However, it can be shown that applying such an assumption on practical lead time may not be valid in case of order crossover. An order crossover occurs when a later order received earlier. This becomes a common phenomenon in many business applications. Any inventory policy based on identically distributed practical lead times needs to be re-investigated. Although the crossover issue has been noticed in the literature, the exact solution for lead-time distribution under crossover remains primitive. Inventory model generally can be separated into two classes, periodic and continuous review. Our paper focuses on continuous-review models. When the lead-time sequence is stationary, some common techniques might be useful to estimate the statistics of the distribution, such as the autocorrelation function and partial autocorrelation function. However, the practical lead time in case of order crossover is not stationary. The proposed method reveals the joint distribution of the practical lead time instead. Any statistics, such as mean and correlation, can be estimated from the joint distribution directly. Subsequently, the optimal inventory police can be obtained in case of order crossover. An exponential distributed lead-time case study is used to demonstrate the use of the proposed method and the risk of mis-use i.i.d. practical lead times. Other phenomena can be studied by similar derivation.

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Exponential distribution; lead time reduction; supply chain management

1. Introduction

A lead time is a time from the initiation to the completion of a process, such as shipping time and manufacturing time. Lead time is concerned not only for customers but also for retailers in many aspects. For example, when the shipping time from supplier to warehouse takes too long, a larger warehouse is needed, thus increase in the storage cost.

Lead time and demand are two most important components in most supply chain systems. Lead time can be basically classified into two types: (1) deterministic lead time means that lead time is fixed, and (2) stochastic lead time means that lead time is a random variable. Similarly, demand can be classified into deterministic and stochastic as well. Table 1 shows those four basic combinations. The first type (Model 1) is with deterministic lead time and deterministic demand. The second type (Model 2) is with deterministic lead time and stochastic demand. These two types of models have deterministic lead times. These models are simpler than models with stochastic lead time. However, any inventory model based on deterministic lead time may not be realistic. The third type (Model 3) is with stochastic lead time and deterministic demand and the fourth one (Model 4) is with stochastic

CONTACT Ching-Chi Yang cuy130@psu.edu

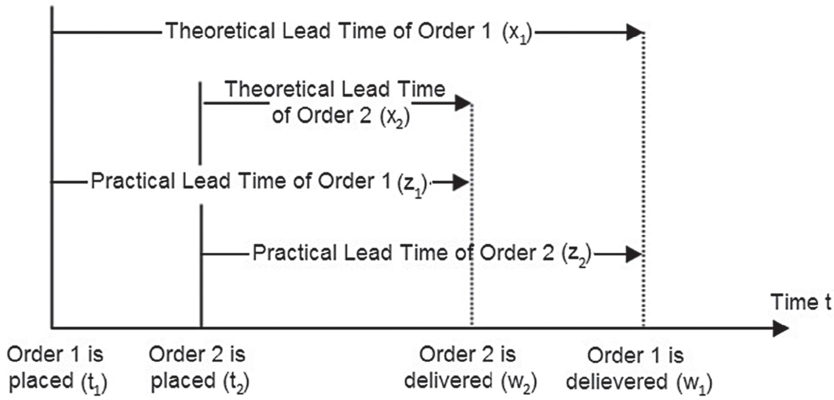


Figure 1. Time line diagram: illustration for order crossover.

Table 1. Classes of supply chain system.

Lead time	Demand rate	
	Deterministic	Stochastic
Deterministic	Model 1	Model 2
Stochastic	Model 3	Model 4

lead time and stochastic demand. The last two types of models focusing on the stochastic lead time are widely used (See, for example, Nasri, Paknejad, & Affisco, 2008; Ryu & Lee, 2003).

Independent and identically distributed (i.i.d.) lead time is assumed in many studies. However, such an i.i.d. assumption is invalid due to order crossover. The order crossover phenomenon is introduced by Hadley and Whitin (1963). It indicates that orders are not received in the same sequence in which they were placed. Take Figure 1 as an example. An order is placed at time t_1 and another order is placed at time t_2 (with $t_1 < t_2$). Suppose order number 1 arrived at time w_1 while order number 2 arrived at time w_2. If w_2 is earlier than w_1, this is called an order crossover. The theoretical lead time of the first order ($x_1 = w_1 - t_1$) is the time from the first order placed (t_1) to the first order delivered (w_1); and the theoretical lead time of the second order ($x_2 = w_2 - t_2$) is the time from the second order placed (t_2) to the second order delivered (w_2). However, when there is a crossover ($w_2 < w_1$), the practical lead time of the first order ($z_1 = w_2 - t_1$) is the time from the first order placed to the second order delivered; while the practical lead time of the second order ($z_2 = w_1 - t_2$) is the time from the second order placed to the first order delivered. Thus, the practical lead time of the first order is shorter than or equal to the theoretical lead time of the first order. Riezebos (2006) distinguishes different types of order crossovers only based on the characteristics of lead time and lead time fluctuations. The literature suggests that order crossovers should not be ignored in modern supply chain management. However, no explicit form has been shown.

Literature analyzes lead time distribution under different review policy: periodic review and continuous review. Under periodic review, Zalkind (1978) provides the distribution of the amount of stock on order while both the demand distribution and the lead time distribution are discrete and bounded. Robinson, Bradley, and Thomas (2001) uses the shortfall distribution to describe the distribution of the practical lead times under order-up-to inventory policies; shows that the inventory cost using the theoretical lead-time demand distribution leads to significantly higher inventory cost. The indicators and methodologies for analyzing, optimizing, and managing the inventory systems should be modified in case of order crossover (See, for example, Bischak, Robb, Silver, & Blackburn, 2013; Bradley & Robinson, 2005; Wensing & Kuhn, 2014).

The continuous-review models are also affected in case of order crossover. Hayya, Bagchi, Kim, and Sun (2008) generates a sequence of realizations of the theoretical lead time, calculate arrival time, sort the arrival time, and then estimate the practical lead time. The mean and autocorrelation of the large sequence are used to represent the overall lead time behavior. Hayya, Harrison, and Chatfield (2009) provides a solution for the intractable inventory model when both demand and lead time are stochastic. The impact of using a sequence of the practical lead time on inventory cost is further studied in Hayya, Harrison, and He (2011). Those studies focus on the fact that the distribution of practical lead time is different from the distribution of theoretical lead time. Estimations, such as mean and variance, are based on 100,000 realizations of the sorted theoretical lead time. Disney, Maltz, Wang, and Warburton (2016) present global logistics data and shows order-up-to policy may not be the cost-optimal policy in global supply chains in case of order crossover. Nielsen, Banaszak, Bocewicz, and Michna (2017) intend to study how often the order crossover occurs and the structure of the practical lead time.

The statistics estimated from a sequence of realizations are used to study the practical lead time structure. The autocorrelation function or partial autocorrelation function are commonly used to present the dependency between two lead times with different lags. However, these estimations from a sequence of realizations suffer from the fact that the practical lead times are not stationary. Especially for earlier orders, mean and variance should be carefully studied since the distributions are different. As can be seen in Figures 3 and 6 later on, the distribution of the first practical lead time is very different from that of the 509th one. Instead of estimating statistics by one sequence, our proposed algorithm estimates statistics for any order and the correlations between any two order's lead time directly. It can be shown that the correlation of any two order's lead times may not be the same even though the time lags of the orders are the same since the practical lead time sequence is a non-stationary process.

The rest of the paper is organized as follows. Section 2 introduces the definitions and procedure to transform the theoretical lead time into the practical lead time. The proposed method is used to study the behavior of the practical lead time. Section 3 provides results under the commonly used exponential distribution. Both theoretical and simulative results are presented. Section 4 illustrates the importance of the practical lead time via a case study. Section 5 concludes advantages and limitations of applying practical lead time in case of order crossover.

2. Problem formulation

Let t_i be the time that the ith order is placed, $i \in \{1, 2, ..., n\}$ and X_i be its corresponding theoretical stochastic lead time (with its realization x_i). Thus the received time is $w_i := t_i + x_i$. When there is no order crossover, we have $w_1 \leq w_2 \leq ... \leq w_n$. When there is order crossover, such an ordering may not hold. We thus sort all w_i's such that $w_{(1)} \leq w_{(2)} \leq ... \leq w_{(n)}$ where $w_{(i)}$ is the i^{th} smallest value among all w_i's. This is the practical received times. Consequently, the practical lead time is $z_i := w_{(i)} - t_i$. Note that, when there is no order crossover, z_i is indeed x_i for all i, but it is not true in case of order crossover. As will be shown, using x_i's in case of order crossover (as in most literature) can be misleading.

Proposed method

The following steps are proposed to study the behavior of the practical lead times,

Step 1. For each order time t, generate a realization x_t from the theoretical lead time, X_t. In this step, n numbers are generated, $(x_1, x_2, ..., x_n)$.

Step 2. Derive their corresponding practical lead times, z_t's. They can be obtained as follow:

(a) Receiving time w_t is $t + x_t$.

(b) Put w_t's in order: $w_{(1)} \leq w_{(2)} \leq ... \leq w_{(n)}$.

(c) The realization of the practical lead time z_t is then $w_{(t)} - t$.

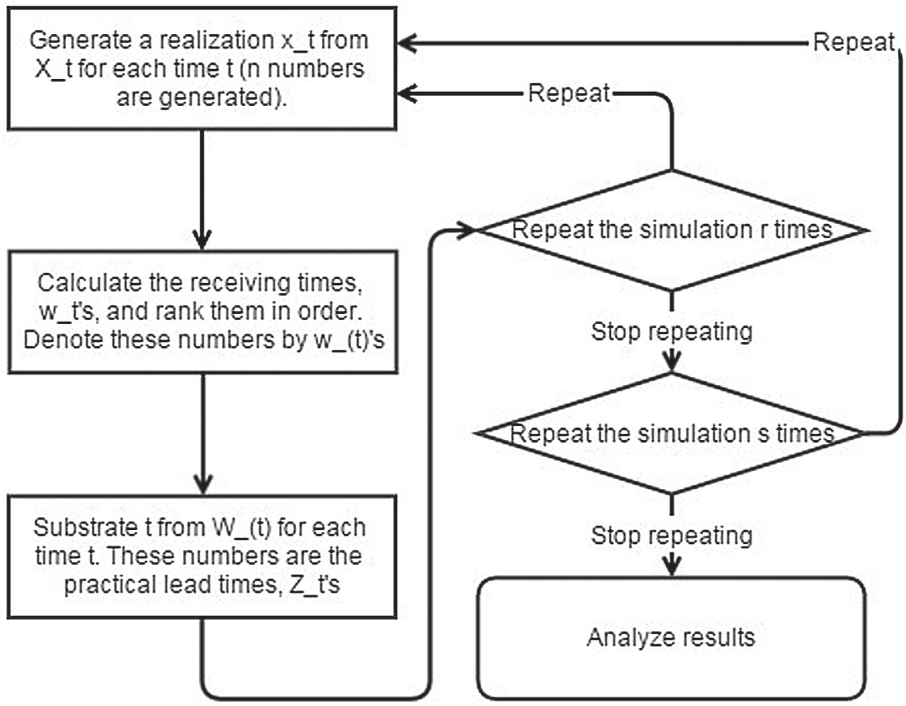


Figure 2. A diagram of the proposed method.

Step 3. Repeat Steps 1 and 2 r times. For each fixed order time t, one empirical distribution based on r realizations is formulated. Density estimation as well as some basic descriptive statistics, such as mean and standard deviation, can be evaluated.

Step 4. Repeat Steps 1 to 3 s times. For each fixed order time t, the end-product is s empirical distributions. One could further study the change of the empirical distribution.

This algorithm first generates a sequence of the theoretical lead time at time t, $t = 1, 2, \ldots, n$. The realizations of the practical lead time are further studied. Thus, by repeating Steps 1 to 2, we collect r realizations of the practical lead time for each fixed time t. An empirical distribution is revealed. It can be shown that practical lead times are neither identically distributed nor independent. For Step 4, we repeat Steps 1 to 3 s times, that is, for each fixed time t, we have s empirical distributions. Then the change of empirical distributions, of means, and of standard deviations can be studied. The procedure can be displayed in Figure 2.

Given any theoretical lead time, the practical lead time can be revealed by the proposed method. Whether there is any order crossover, the practical lead time is always valid. However, the theoretical derivation for the practical lead time distribution is only possible for some specific case.

3. Exponentially distributed lead time

One of the popularly used distributions of theoretical lead times is the exponential distribution. This section provides (theoretical as well as empirical) results of the practical lead time under exponentially distributed theoretical lead time. Section 3.1 studies the first practical lead time (Z_1) and finds that it (Z_1) is very different from the first theoretical lead time (X_1). Section 3.2 provides applications of the proposed method. In Section 3.3, statistics, such as mean, variance, and correlation, of the practical lead time are summarized when the mean of the exponential distributed theoretical lead time is 10 (λ

= 1/10). These results show that (1) the practical lead time is different from the theoretical lead time, and (2) the practical lead times are not identically distributed. In Section 3.4, it can be further shown such a phenomenon is held when changing the value of parameter λ; that is, the practical lead time is generally different from the theoretical lead time due to order crossover.

3.1. Theoretical properties of the first practical lead time (Z_1)

Suppose X_1 and X_2 are independent continuous random variables. Let Z be $\min\{X_1, X_2\}$. The cumulative distribution function of random variable Z is,

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(\min\{X_1, X_2\} \le z) \\ &= P(X_1 \le z) + P(X_2 \le z) - P(X_1 \le z,\, X_2 \le z) \\ &= P(X_1 \le z) + P(X_2 \le z) - P(X_1 \le z)P(X_2 \le z). \end{aligned}$$

The probability density function of Z is thus,

$$\begin{aligned} f_Z(z) &= f_{X_1}(z) + f_{X_2}(z) - f_{X_1}(z)P(X_2 \le z) - P(X_1 \le z)f_{X_2}(z) \\ &= f_{X_1}(z)[1 - P(X_2 \le z)] + f_{X_2}(z)[1 - P(X_1 \le z)] \\ &= f_{X_1}(z)\int_z^\infty f_{X_2}(x)\mathrm{d}x + f_{X_2}(z)\int_z^\infty f_{X_1}(x)\mathrm{d}x. \end{aligned}$$

Hence, we have Proposition 3.1 below.

Proposition 3.1: *If X_1 and X_2 are independent continuous random variables and $Z := \min\{X_1, X_2\}$, then*

$$f_Z(z) = f_{X_1}(z)\int_z^\infty f_{X_2}(x)dx + f_{X_2}(z)\int_z^\infty f_{X_1}(x)dx.$$

Using Proposition 3.1, the distribution of practical lead time Z_1 can be generated when the distribution of theoretical lead times are given. For instance, if the theoretical lead times are i.i.d. exponentially distributed, the first practical lead time, Z_1, can be shown in Theorem 3.1. The proof of Theorem 3.1 is shown in Appendix 1.

Theorem 3.1: *Let an order be placed at each time, $1, 2, \ldots,$ and n. If the theoretical lead times follow an exponential distribution, $f(t) = \lambda\exp(-\lambda t)$, then the distribution of the first practical lead time, $Z_1^{(n)}$, is as follows,*

$$f_{Z_1^{(n)}}(t) = \begin{cases} \lambda\exp(-\lambda t), & \text{if } 0 \le t \le 1; \\ 2\lambda\exp(-\lambda(2t-1)), & \text{if } 1 \le t \le 2; \\ 3\lambda\exp(-\lambda(3t-3)), & \text{if } 2 \le t \le 3; \\ \cdots & \\ (n-1)\lambda\exp\left(-\lambda\left((n-1)t - \frac{(n-2)(n-1)}{2}\right)\right), & \text{if } n-2 \le t \le n-1; \\ n\lambda\exp\left(-\lambda\left(nt - \frac{(n-1)n}{2}\right)\right), & \text{if } n-1 \le t. \end{cases}$$

Figure 3 shows the pdf of the theoretical and the practical lead time (taking $\lambda = 0.1$ as an example). These two distributions are clearly different. For example, around 90% probability that the practical lead time occurs within a certain period, say [0, 7), but only around 50% probability that the theoretical lead does. Precisely, the probability of the practical lead time less than 7 is 0.9392; and the theoretical lead time is less than 7 is 0.5034. More general results will be shown in Section 3.4.

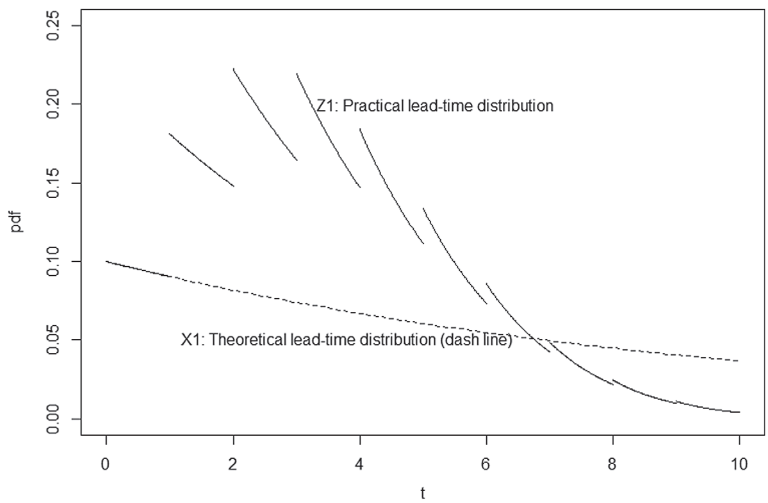


Figure 3. Probability density functions, $f_{Z_1}(t)$ and $f_{X_1}(t)$, given $\lambda = 0.1$.

3.2. Distribution of the practical lead time in general

The practical lead time for other time period, in general, is rather complicated. Based on the proposed method in Section 2, realizations of the practical lead times are constructed. Thus, the empirical distribution of the practical lead times and the correlations among the practical lead times can be obtained.

Assume the theoretical lead times, X_1, X_2, ..., and X_n, are i.i.d. exponential distributed, that is, $f_{X_i}(x) = \lambda \exp(-\lambda x)$. Suppose an order is placed at each time t ($t = 1, 2, ..., 1000$). The proposed method generates the realizations of the theoretical lead times, x_t's, and then reveals the realization of the practical lead times. By repeating Steps 1 and 2 of the proposed method 500 times, we can estimate the density by the empirical distribution of practical lead time of the order at any fixed time t. Its statistics, such as mean and standard deviation, can also be calculated. Repeating the procedure of gathering the empirical distribution at any fixed time t 100 times, we can study the change of empirical distribution and the changes of mean, standard deviation, and correlation through the time t.

3.3. Scenario: $\lambda = 0.1$, $r = 500$, $s = 100$, and $n = 1000$

As an example, this Section studies the behavior of all practical lead times (Z_1, Z_2, ..., and Z_n) when $\lambda = 0.1$. The number $r = 500$ controls the number of realizations of the practical lead time for each time t; while the number $s = 100$ controls the number of empirical distributions for each fixed time t. Also set $n = 1000$.

Figure 4 shows the mean practical lead times for the first 20 orders. The x-axis in Figure 4 indicates the time that order is placed. The mean lead time for the first order is 3.48 which is consistent with Theorem 3.1 (Section 3.1). The mean lead time for each order increases as the time increases until the mean lead time approach to 10. This fact indicates that the practical lead times are not identically distributed.

However, for the practical lead time of orders placed at time t where t is large, the marginal distributions are similar. Figure 5 shows the box plot of the mean of each 500 realizations when the time order placed at 100, 200, ..., and 900. Take the time order placed at time 900 as an example, under the scenario $\lambda = 0.1$, the probability that the theoretical lead time (X_{900}) is greater than 100 is less than 0.001, i.e. $P(X_{900} > 100) < 0.001$. In other words, the orders which are placed after the

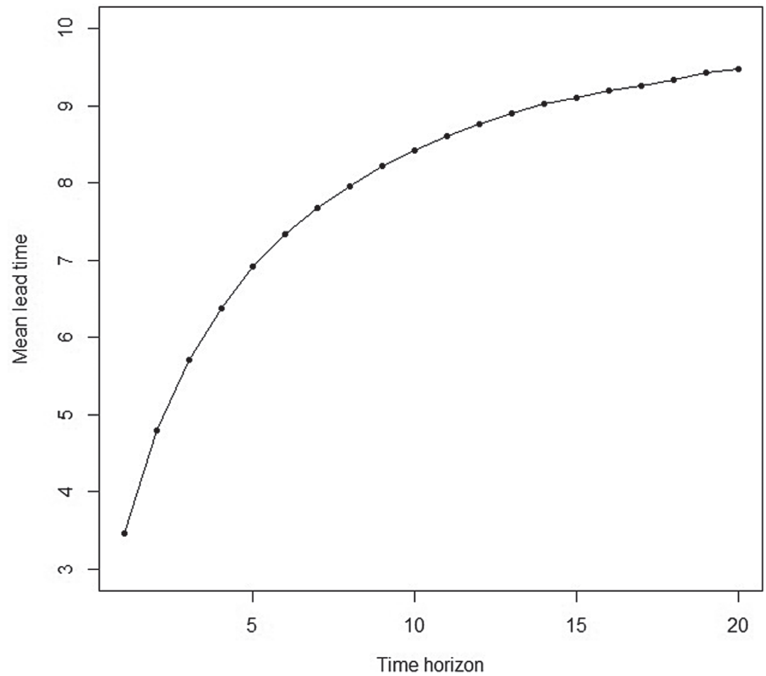


Figure 4. Mean practical lead time.

time 1000 (= 900 + 100) will seldom affect the practical lead time of the order placed at time 900. The impact of the orders placed after time 1000 can be ignored when we are only interested in the practical lead time of orders placed before time 900. Thus, the estimations in Figure 5 reflect the true estimations of the practical lead times. The long-term mean practical lead time is around 10 (between 9.35 and 10.35). The standard error of means for each ordering time is within 0.091 and 0.103. The long-term mean practical lead times, therefore, may be treated as a constant, 10. If the long-term mean practical lead time is used to replace all mean practical lead times (the fact that practical lead times are not identically distributed is ignored), the mean practical lead time will be treated as the same as the mean theoretical lead time. Subsequently, the estimations of any inventory model which is involved with mean lead time, such as inventory cost, will be the same. However, the practical lead times are not identically distributed. The mean practical lead times are not the same. Any model involved with mean lead time should be re-investigated in the case of order crossover.

Take $t = 509$ as a typical example. The realizations of the practical lead time can be used to indicate that the practical lead time is no longer exponential distributed. Figure 6 shows a histogram of the realizations of the practical lead time at $t = 509$, the probability density function of an exponential distribution with parameter $\lambda = 0.1$ (the curve in Figure 6(a)), and the probability density function of a gamma distribution with parameter $\alpha = 20$ and $\beta = 2$ (the curve in Figure 6(b)). The histogram of Z_{509} in Figure 6(a) shows clearly that its realizations do not follow an exponential distribution. The vertical line of Figure 6(a) is the 99% quantile of 500 realizations of the practical lead time. The probability of the theoretical exponential lead time less than the vertical line is less than 80% (shaded area in Figure 6(a)). In other words, with probability 20% or more, the model using the theoretical lead time is improper. The sample standard deviation of the practical lead time can be estimated as well. The sample standard deviation is around 2.23 ($\approx \sqrt{1/(2\lambda)} = 2.236$) while the population standard deviation of the theoretical lead time is 10. In general, the distribution of all practical lead times do not follow the distribution of the theoretical lead time.

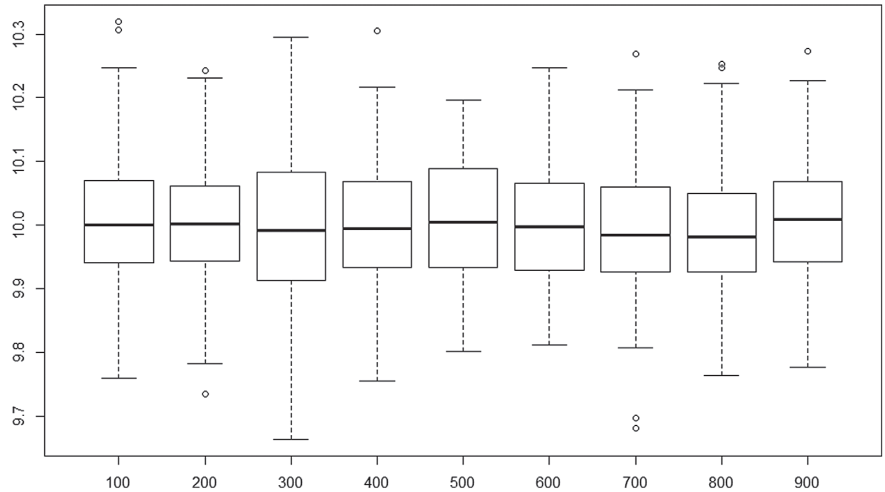
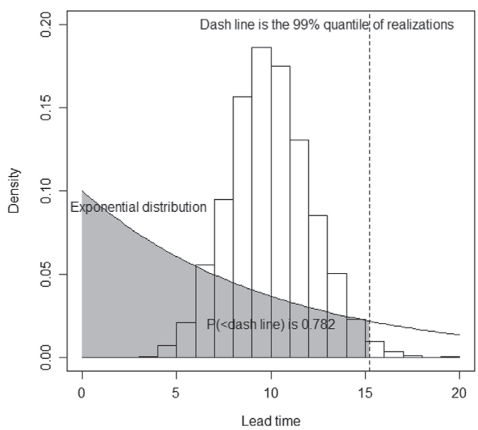
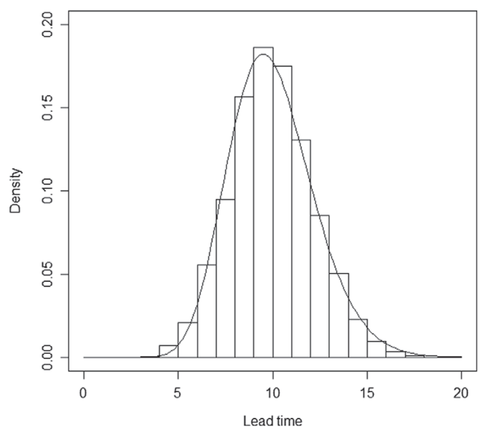


Figure 5. Box plot of means of practical lead times.



(a) Theoretical lead time.

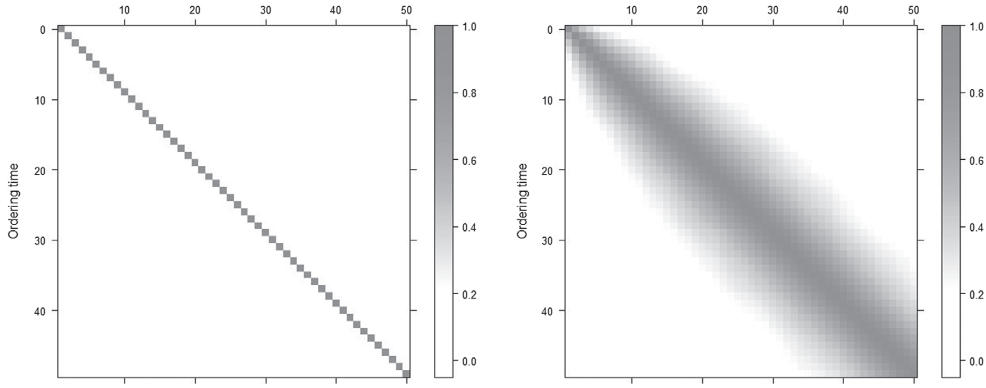


(b) Gamma distribution.

Figure 6. Histogram of a set of realizations for $t = 509$.

In fact, Figure 6(b) shows that the practical lead time at $t = 509$ behaves like a gamma distribution $f_Z(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}$, ($\alpha = 20$ and $\beta = 2$) with mean 10 and standard deviation $\sqrt{5}$. On the other hand, Theorem 3.1 provides the mean and standard deviation of the first practical lead time, 3.48 and 2.06, respectively. Regardless, the dependence of the lead times, the practical lead times are not identically distributed and different from the theoretical lead times.

If not independent, the dependence of the practical lead times cannot be ignored. Correlation is a basic estimation of referring to any departure of the practical lead times from independence. The correlations among the theoretical lead times will be a baseline for highlighting the relatively high correlation among the practical lead times. Figure 7 shows the correlations among the first 50 practical lead times (Figure 7(a)) and the correlations among the first 50 theoretical lead times (Figure 7(b)). Each graph contains 50×50 cells. The scale of the cell for the cth column and the rth row represents the level of correlation between the cth lead time and the rth lead time. A light scale indicates that two lead times are less correlated. Since the theoretical lead times are generated from



(a) Theoretical lead time. (b) Practical lead time.

Figure 7. Correlations among the first 50 lead times.

i.i.d. exponential distribution, the correlations among the theoretical lead times are low, as can be seen in Figure 7(a). However, the correlations among the practical lead times are relatively high as shown in Figure 7(b). Specifically, the maximum of the correlations from the first 50 theoretical lead time is 0.15; while 35% of the correlations from the first 50 practical lead time is greater than 0.15. Thus, the independence assumption cannot be applied on the practical lead time in case of order crossover. Moreover, the correlations are not the same even though the lags of two orders are the same. Let's use the correlations between lead times at the 1st order and the 5th order and at the 21st order and the 25th order as an example. The correlation between lead times at the 1st order and the 5th order is around 0.27 while the one at the 21st order and the 25th order is around 0.66. Using one number for the lead times with the same lag, therefore, might not be appropriate.

3.4. General λ values

In general, if the theoretical lead times follow an exponential distribution with parameter λ, the long-term mean practical lead time is close to the expectation of the theoretical lead time, $1/\lambda$. The standard deviation of the practical lead time and that of the theoretical lead time behavior differently. Table 2 shows the standard deviation (SD) of the practical lead time, the value $\sqrt{1/(2\lambda)}$, and the standard deviation of the corresponding exponential distribution when $\lambda = 2, 1, 1/5, 1/10, 1/20$, and $1/50$. It indicates the standard deviation of the practical lead time is close to $\sqrt{1/(2\lambda)}$ instead of the standard deviation of the theoretical lead time, $1/\lambda$, when λ is small.

The fact that the standard deviation of the practical lead time differs from that, $1/\lambda$, of the exponential distribution can basically show that the practical lead times do not follow the exponential distribution. Note that, a gamma distribution with $\alpha = \frac{2}{\lambda}$ and $\beta = 2$ will have mean $1/\lambda$ and standard deviation $\sqrt{1/(2\lambda)}$. It can be used to estimate the empirical distribution of the practical lead time. Thus, in case of order crossover, the standard deviation of practical lead time is less than $\min\{1/\lambda, \sqrt{1/(2\lambda)}\}$. It is also shown that different frequency of order crossover leads to different behavior of the practical lead times even though the theoretical lead times follows the exponential distributions.

4. Case study

The Economic-Order-Quantity model (EOQ) is a result of classical optimization of inventory related costs. Nasri, Paknejad, and Affisco (2012) analyze the flexibility and quality improvement in a quality-adjusted EOQ model with finite-range stochastic lead-time. The flexibility improvement

Table 2. Simulations by different lambda.

	λ					
	2	1	1/5	1/10	1/20	1/50
SD	0.42	0.65	1.56	2.22	3.14	4.98
$\sqrt{1/(2\lambda)}$	0.50	0.71	1.58	2.24	3.16	5.00
$1/\lambda$	0.50	1.00	5.00	10.00	20.00	50.00

model reduces almost 10% of the optimal average cost per unit time in the quality-adjusted EOQ model. However, the variability of lead time could be an issue in the case of order crossover (as shown in Section 3).

One of the main focuses of EOQ models is to predict the expected average cost (EAC) per unit time. When the predicted cost is greater than the real cost, a company's budget is overestimated. On the other hand, if the predicted cost is less than the real cost, the shortage will cause seriously short of liquidity. The optimal average cost in the EOQ model is thus important to avoid these shortages. The accurate estimation of the optimal average cost is desirable.

The basic EOQ model developed by Sphicas and Nasri (1984) is a stochastic lead time model with mean μ, variance V and support in (α, β). The optimal expected average cost per unit time, EAC_{basic}, can be obtained as

$$EAC_{\text{basic}} = \sqrt{[2DK + VD^2(h+p)]/\left(\frac{1}{h} + \frac{1}{p}\right)},$$

where, D = demand per unit time (in units),
K = setup cost per setup,
h = holding cost per (non-defective) unit per unit time,
p = backorder cost per (non-defective) unit cost per unit time, and
V = variance of the lead time.

A similar model involved with defective units is proposed Paknejad, Nasri, and Affisco (2005). The optimal expected average cost per unit time of the model, EAC_{adj}, is obtained as $\frac{h}{2}\frac{\rho}{1+\rho} + \eta EAC_{\text{basic}}$ where $\eta = \left[1 + 2h'\rho\left(\frac{1}{h} + \frac{1}{p}\right)\right]^{1/2}$, ρ is the ratio of the probability of a defective unit to of a nondefective unit, and h' is defective holding cost per unit per unit time. The same restriction is applied to both models. The restriction is used to guarantee that orders do not cross, see Appendix 2 for details. The restriction results in a small range of the stochastic lead time. However, orders do cross over and the small range of the stochastic lead time is unrealistic.

To study the impact of the EAC using the practical lead time and the theoretical lead time, a thorough simulation is conducted. Set $D = 520$, $K = 50$, $h = 10$, $p = 20$, $\rho = 0.25$, and $\lambda = 1$. Since the variance of the lead time affects the optimal expected average cost, the proposed algorithm is used to estimate the variance of the practical lead time using $\lambda = 1$. The variances of the practical lead times are from 0.38 to 0.42. Note that a smaller variance leads to a smaller value of the EOQ model. The largest variance of the practical lead time is used as the baseline. Table 3 shows the optimal expected average cost calculated by applying the theoretical and the practical lead time for both EOQ and adjusted EOQ model. Under basic EOQ model, the difference between the EAC formula with the theoretical lead time and with the practical lead time is 2591.22 (7377.44–4786.22). This difference is 35% of the EAC under the theoretical lead time. It is a great proportion of the EAC of the EOQ model with the theoretical lead time. Similarly, under adjusted EOQ model, the difference between the EAC of adjusted EOQ model with the theoretical lead time and with the practical lead time is 3038.48

Table 3. Predictions based on the theoretical and practical lead times.

	Theoretical LT	Practical LT
EAC	7377.44	4786.22
EAC_{adj}	8650.82	5612.34

(8650.82–5612.34), a 35% of the EAC with the theoretical lead time. It is a great proportion of the EAC of the adjusted EOQ mode with the theoretical lead time as well. The outcomes are different. The estimation from the practical lead time reflects the reality; the estimation from the theoretical lead time is larger than the real cost. As the previous discussion, the company's budget is overestimated if the theoretical lead time is used.

5. Conclusion

Lead time plays an important role in many areas, such as supply chain, economics, and marketing. For stochastic lead times, the distribution of lead time, which describes the uncertainty of lead time, is thus critical. The conventional assumption is that the lead times are independent and identically distributed (i.i.d.). However, such an assumption may not be valid due to order crossover - a crossover is a case when orders are not received in the same sequence in which they were placed. Although order crossover phenomenon has been noticed by Hadley and Whitin (1963), their theoretical/practical properties remain primitive. Recently, order crossover has received more attention as it has become a common phenomenon in this e-business era.

In the case of order crossover, the distribution of the practical and theoretical lead time may be very different. The joint distribution of the practical lead times can be revealed by the proposed method. The marginal distribution of the practical lead times are shown to be non-stationary (clearly not i.i.d.). The descriptive statistics of any practical lead time at different order time should be estimated separately. Consequently, any inventory policy based on stationary lead times may need to be re-investigated due to order crossover. Take the EOQ model as an example, the distribution of the practical lead time can be obtained via the proposed method. The expected average cost based on the practical lead time is more reliable than that based on the theoretical lead time, leading to a different inventory policy.

Different frequency of the order crossovers leads to different behavior of the practical lead times as well. The four-step proposed method provides a solution to describe the joint distribution of the practical lead time for any stochastic lead-time model due to order crossover. Although the method requires numerical simulation to yield an empirical joint distribution of the practical lead times, the consistency of the joint distributions can be evaluated (Step 4). In fact, the proposed method can be applied to any distribution for theoretical lead time.

ORCID

Ching-Chi Yang http://orcid.org/0000-0003-2517-2709

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Appendix 1. Proof of theorem 3.1

Mathematical induction is used to prove the proposed function holds for any n. The first step shows that the function holds when $n = 1$. The second step shows, if the function holds when $n = k$, the function holds when $n = k + 1$. By induction, the function is true for any n. The steps shows as follows,

1°: For $n = 1, f_{Z_1^{(1)}}(t) = f_{X_1}(t) = \lambda \exp(-\lambda t)$, where $0 \le t$. The function holds.

2°: if $n = k$ is true, that is,

$$f_{Z_1^{(k)}}(t) = \begin{cases} \lambda \exp(-\lambda t), & \text{if } 0 \le t \le 1; \\ 2\lambda \exp(-\lambda(2t-1)), & \text{if } 1 \le t \le 2; \\ 3\lambda \exp(-\lambda(3t-3)), & \text{if } 2 \le t \le 3; \\ \cdots \\ (k-1)\lambda \exp\left(-\lambda\left((k-1)t - \frac{(k-2)(k-1)}{2}\right)\right), & \text{if } k-2 \le t \le k-1; \\ k\lambda \exp\left(-\lambda\left(kt - \frac{(k-1)k}{2}\right)\right), & \text{if } k-1 \le t; \end{cases}$$

For $n = k + 1, Z_1^{(k+1)} = \min\{Z_1^{(k)} + 1, X_{k+1} + (k+1)\} - 1 = \min\{Z_1^{(k)}, X_{k+1} + k\}$. By Proposition 1, when $t \ge k$,

$$\begin{aligned} f_{Z_1^{(k+1)}}(t) &= k\lambda \exp\left(-\lambda\left(kt - \frac{(k-1)k}{2}\right)\right)\int_{t-k}^{\infty} \lambda \exp(-\lambda s)\mathrm{d}s \\ &\quad + \lambda \exp(-\lambda(t-k))\int_{t}^{\infty} k\lambda \exp\left(-\lambda\left(ks - \frac{(k-1)k}{2}\right)\right)\mathrm{d}s \\ &= k\lambda \exp\left(-\lambda\left(kt - \frac{(k-1)k}{2}\right)\right)\exp(-\lambda(t-k)) \\ &\quad + \lambda \exp(-\lambda(t-k))\exp\left(-\lambda\left(kt - \frac{(k-1)k}{2}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= k\lambda \exp\left(-\lambda(k+1)t - k\left(\frac{k-1}{2}+1\right)\right)\\
&\quad + \lambda \exp\left(-\lambda(k+1)t - k\left(\frac{k-1}{2}+1\right)\right)\\
&= (k+1)\lambda \exp\left(-\lambda(k+1)t - \left(\frac{k(k+1)}{2}\right)\right)
\end{aligned}
$$

Thus,

$$
f_{Z_1^{(k+1)}}(t) = \begin{cases}
\lambda \exp(-\lambda t), & \text{if } 0 \le t \le 1;\\
2\lambda \exp(-\lambda(2t-1)), & \text{if } 1 \le t \le 2;\\
3\lambda \exp(-\lambda(3t-3)), & \text{if } 2 \le t \le 3;\\
\cdots & \\
k\lambda \exp\left(-\lambda\left(kt - \frac{(k-1)k}{2}\right)\right), & \text{if } k-1 \le t \le k;\\
(k+1)\lambda \exp\left(-\lambda\left((k+1)t - \frac{k(k+1)}{2}\right)\right), & \text{if } k \le t;
\end{cases}
$$

Then $n = k + 1$ is also true.

Appendix 2. Restriction of the EOQ models

The basic model Sphicas and Nasri (1984) and the model involving defective units Paknejad et al. (2005) have the same restriction in order to have no crossover. Notice that these two models did not require independency of lead times and assume the lead times is uniform distributed, denoted by $U(\alpha, \beta)$. The restriction is $k \ge k_2$ where

$$
\begin{aligned}
k &= \frac{2K}{(h+p)D},\\
k_2 &= \frac{(\mu-\alpha)^2}{\Omega} - V, \quad \text{if } \Omega \le \frac{\mu-\alpha}{\beta-\mu}\\
k_2 &= \Omega(\mu-\beta)^2 - V, \quad \text{if } \Omega \ge \frac{\mu-\alpha}{\beta-\mu}\\
\Omega &= h/p.
\end{aligned}
$$