Statistica Sinica **26** (2016), 589-603
doi:http://dx.doi.org/10.5705/ss.2013.352

CONSTRUCTION OF SLICED MAXIMIN-ORTHOGONAL LATIN HYPERCUBE DESIGNS

Jinyu Yang[1], Hao Chen[2], Dennis K. J. Lin[3] and Min-Qian Liu[1]

[1]*Nankai University,* [2]*Tianjin University of Finance and Economics and* [3]*The Pennsylvania State University*

Abstract: A sliced Latin hypercube design is a special Latin hypercube design that can be divided into slices of smaller Latin hypercube designs. This type of designs is useful for computer experiments with qualitative and quantitative factors, multiple experiments, data pooling, and cross-validation. Orthogonality and uniformity are important properties for Latin hypercube designs. In this paper, sliced maximin-orthogonal Latin hypercube designs are constructed using orthogonal designs, Goethals-Seidel arrays, and Kharaghani arrays. The resulting designs have both second-order orthogonality and good uniformity.

Key words and phrases: Computer experiment, maximin distance, orthogonality, sliced Latin hypercube design.

1. Introduction

Sliced Latin hypercube designs (LHDs), first introduced by Qian (2012), are useful for computer experiments with qualitative and quantitative factors, multiple computer experimentations, data pooling, and cross-validation. Such a design is a special type of LHD, which can be divided into slices of smaller LHDs. Orthogonality and uniformity are two desirable properties for LHDs. An LHD is orthogonal if the inner product of any two columns is zero. It is second-order orthogonal if both the inner product of any two columns and the sum of the elementwise products of any three columns are zero, desirable when quadratic effects and bilinear interactions are present. Such a design guarantees that the estimates of all linear effects are uncorrelated with each other, and with that of all quadratic effects and bilinear interactions (cf., Ye (1998), Sun, Liu, and Lin (2010), Yang and Liu (2012)). Construction of orthogonal and nearly orthogonal LHDs has been actively studied, including Ye (1998), Steinberg and Lin (2006), Cioppa and Lucas (2007), Georgiou (2009), Pang, Liu, and Lin (2009), Lin, Mukerjee, and Tang (2009), Bingham, Sitter, and Tang (2009), Sun, Liu, and Lin (2009, 2010), Lin et al. (2010), Georgiou and Stylianou (2011), Yang and Liu (2012), Ai, He, and Liu (2012), and Georgiou and Efthimiou (2014), among others. Apart from the orthogonality, experimenters are also interested in spreading the points evenly across the experimental region. Johnson, Moore, and Ylvisaker (1990) introduced the maximin distance criterion, which maximizes the minimum inter-site distance. Morris and Mitchell (1995) applied this criterion for finding optimal LHDs.

The sliced LHDs proposed by Qian (2012) can only achieve a one-dimensional uniformity, but not orthogonality. Recently, Yang et al. (2013) and Huang, Yang, and Liu (2013) proposed methods to construct sliced LHDs with orthogonality or near orthogonality, but not good uniformity; Based on (resolvable) orthogonal arrays, Yin, Lin, and Liu (2014) and Yang, Chen, and Liu (2014) constructed sliced LHDs with an attractive low-dimensional uniformity, but not orthogonality. In this paper, we propose a new class of sliced LHDs, called sliced maximin-orthogonal LHDs which are sliced orthogonal LHDs with good uniformity. Three construction methods for sliced orthogonal LHDs are presented. Then through an optimal column-exchange algorithm, sliced maximin-orthogonal LHDs from the resulting sliced orthogonal LHDs are obtained. The number of slices of the constructed designs is rather flexible.

The remainder of this paper is organized as follows. Section 2 presents the construction of sliced orthogonal LHDs. The sliced maximin-orthogonal LHDs are obtained in Section 3. Properties of the resulting designs are discussed in Section 4. All proofs are deferred to the Appendix.

2. Construction of Sliced Orthogonal LHDs

This section provides three methods for constructing sliced orthogonal LHDs with k slices, where k is a positive integer. We first introduce some relevant definitions and notations. An LHD with n runs and m factors, denoted by $L(n,m)$, has each column composed of n equally-spaced levels. For convenience, the n levels are taken to be $\{-(n-1),-(n-3),\ldots,(n-1)\}$. An $L(n,m)$ with $n=qk$ is called a sliced LHD, if it can be divided into k slices each of which becomes an $L(q,m)$ when the n levels are collapsed to q equally-spaced levels according to:

$$
\begin{aligned}
&\{-(n-1),-(n-3),\ldots,-(n-2k+1)\}\rightarrow-(q-1),\\
&\{-(n-2k-1),-(n-2k-3),\ldots,-(n-4k+1)\}\rightarrow-(q-3),\ldots,\\
&\{n-2k+1,n-2k+3,\ldots,n-1\}\rightarrow q-1.
\end{aligned}
$$

Hence, the q levels of each slice correspond to the q equally-spaced intervals $[-n,-n+2k),[-n+2k,-n+4k),\ldots,[n-4k,n-2k),[n-2k,n)$. A sliced $L(n,m)$ with $n=qk$ is called orthogonal, denoted by $SOL(n,m,k)$, if the whole design is orthogonal as well as design within each slice.

Suppose $S = \{S_i : S_i = (s_{i0}, s_{i1}, \ldots, s_{i(n-1)}), i = 1, \ldots, k\}$ is a set of k vectors of length n. The t *periodic autocorrelation function* (PAF) of S is defined as

$$P_S(t) = \sum_{i=1}^{k} P_{S_i}(t) = \sum_{i=1}^{k} \sum_{j=0}^{n-1} s_{ij} s_{i,j+t}, \quad t = 0, \ldots, n-1,$$

where $j + t$ equals $(j + t) \mod n$ when $j + t \geq n$. S is said to have a zero PAF if $P_S(t) = 0$, for any $t = 1, \ldots, n-1$.

Suppose v is a vector of order m, the circulant matrix with v as the first row is a square matrix of order m, denoted by $C(v)$, where each row vector is rotated one element to the right relative to the preceding row vector.

2.1. Construction using orthogonal designs

In this subsection, sliced orthogonal LHDs are constructed via a special type of orthogonal design (OD) (see Yang and Liu (2012)).

Definition 1 (Yang and Liu (2012))**.** An $m \times m$ matrix D is an m-order OD with entries from $\pm(ia+b)$ for $i = 0, \ldots, m-1$ and $a \neq 0$, denoted by $OD(m)$, if it satisfies the properties: by changing $-(ia+b)$ to $ia+b$ (for $i = 0, \ldots, m-1$) in D, each column is a permutation of $\{ia + b, i = 0, \ldots, m-1\}$; the inner product of any two distinct columns is zero.

$OD(m)$'s are available in Yang and Liu (2012) for $m = 2^r$, where r is any positive integer. A new construction method for sliced orthogonal LHDs with k slices can be obtained by the following algorithm. It is able to construct an $SOL(k \cdot 2^{r+1}, 2^r, k)$ for any positive integer r, where the factor number is 2^r. New sliced LHDs (with other numbers of factors) will be presented.

Algorithm 1.

Step 1. Construct an $OD(2^r)$, denoted by D_{ab}, with elements of the form $\pm(ja+b)$ for $j = 0, \ldots, 2^r - 1$, where $a \neq 0$.

Step 2. Set $L_i = (D^T_{2k,2i-1}, -D^T_{2k,2i-1})^T$, for $i = 1, \ldots, k$.

Step 3. The final design L is obtained by stacking $L_1, \ldots, L_k$ row by row, $L = (L_1^T, \ldots, L_k^T)^T$.

Theorem 1. *An L constructed by Algorithm* 1 *is an $SOL(2km, m, k)$ with k slices $L_1, \ldots, L_k$, where $m = 2^r$ and L_i is an orthogonal $L(2m, m)$. When projected onto each dimension, each of the* $2m$ *equally-spaced intervals* $[-2km, -2k(m-1)), [-2k(m-1), -2k(m-2)), \ldots, [-2k, 0)$, $[0, 2k), \ldots, [2k(m-1), 2km)$ *contains exactly one point of each slice.*

An example is given to illustrate the construction.

Example 1. Let $r = k = 2$, then $m = 2^r = 4$, $n = 2km = 16$. Following Step 1, construct the $OD(4)$ with

$$D_{ab} = \begin{pmatrix} b & a+b & -3a-b & 2a+b \\ a+b & -b & -2a-b & -3a-b \\ 2a+b & 3a+b & a+b & -b \\ 3a+b & -2a-b & b & a+b \end{pmatrix}.$$

Take $a = 4$, and for $b = 1, 3$, we have

$$D_{41} = \begin{pmatrix} 1 & 5 & -13 & 9 \\ 5 & -1 & -9 & -13 \\ 9 & 13 & 5 & -1 \\ 13 & -9 & 1 & 5 \end{pmatrix} \text{ and } D_{43} = \begin{pmatrix} 3 & 7 & -15 & 11 \\ 7 & -3 & -11 & -15 \\ 11 & 15 & 7 & -3 \\ 15 & -11 & 3 & 7 \end{pmatrix}.$$

Let $L = (D_{41}^T, -D_{41}^T, D_{43}^T, -D_{43}^T)^T$, then L is an $SOL(16, 4, 2)$ with 2 slices: $L_1 = (D_{41}^T, -D_{41}^T)^T$ and $L_2 = (D_{43}^T, -D_{43}^T)^T$, where L_1 and L_2 are two orthogonal $L(8, 4)$'s, and

$$\begin{aligned} L &= (L_1^T, L_2^T)^T \\ &= \left(\begin{array}{cccccccc|cccccccc} 1 & 5 & 9 & 13 & -1 & -5 & -9 & -13 & 3 & 7 & 11 & 15 & -3 & -7 & -11 & -15 \\ 5 & -1 & 13 & -9 & -5 & 1 & -13 & 9 & 7 & -3 & 15 & -11 & -7 & 3 & -15 & 11 \\ -13 & -9 & 5 & 1 & 13 & 9 & -5 & -1 & -15 & -11 & 7 & 3 & 15 & 11 & -7 & -3 \\ 9 & -13 & -1 & 5 & -9 & 13 & 1 & -5 & 11 & -15 & -3 & 7 & -11 & 15 & 3 & -7 \end{array}\right)^T. \end{aligned}$$

2.2. Construction using Goethals-Seidel arrays

We next propose a method to construct sliced orthogonal LHDs by applying the Goethals-Seidel arrays. A Goethals-Seidel array is of the form

$$GS(S_1, S_2, S_3, S_4) = \begin{pmatrix} S_1 & S_2R_p & S_3R_p & S_4R_p \\ -S_2R_p & S_1 & -S_4^TR_p & S_3^TR_p \\ -S_3R_p & S_4^TR_p & S_1 & -S_2^TR_p \\ -S_4R_p & -S_3^TR_p & S_2^TR_p & S_1 \end{pmatrix},$$

where R_p is the back-diagonal identity matrix of order p, and S_1, S_2, S_3, and S_4 are circulant matrices of order p (Goethals and Seidel (1967)).

Lemma 1 (Geramita and Seberry (1979), Thm. 4.49)**.** *If the circulant matrices S_1, S_2, S_3, and S_4 satisfy*

$$S_1S_1^T + S_2S_2^T + S_3S_3^T + S_4S_4^T = fI_p,$$

then $GS(S_1, S_2, S_3, S_4)$ is an orthogonal matrix of order $4p$.

Corollary 1. *If there exist vectors S_1, S_2, S_3, S_4 of length p with zero PAF, then they can be used as the first rows of circulant matrices that can be placed in the Goethals-Seidel array to obtain an orthogonal matrix.*

The key issue here is to obtain the sliced structure by constructing appropriate Goethals-Seidel arrays.

Algorithm 2.

Step 1. Find vectors v_1, v_2, v_3, and v_4 of the same length p with zero PAF, where the elements come from the set $\{b, b+a, \ldots, b+(4p-1)a\}$ if the signs are ignored.

Step 2. Place the circulant matrices $C(v_1)$, $C(v_2)$, $C(v_3)$, and $C(v_4)$ into a Goethals-Seidel array to obtain $G_{ab} = GS(C(v_1), C(v_2), C(v_3), C(v_4))$.

Step 3. For a given k, set $L_i = (G^T_{2k,2i-1}, -G^T_{2k,2i-1})^T$, for $i = 1, \ldots, k$. Construct L by stacking $L_1, \ldots, L_k$ row by row, $L = (L_1^T, \ldots, L_k^T)^T$.

Theorem 2. *An L constructed by Algorithm 2 is an $SOL(8kp, 4p, k)$ with k slices $L_1, \ldots, L_k$, where L_i is an orthogonal $L(8p, 4p)$. When projected onto each dimension, each of the $8p$ equally-spaced intervals $[-8pk, -(8pk-2k)), \ldots, [-2k, 0)$, $[0, 2k), \ldots, [8pk-2k, 8pk)$ contains exactly one point of each slice.*

Example 2. With $k = p = 3$, an $SOL(72, 12, 3)$ can be obtained where each slice is an $L(24, 12)$. Take the vectors

$$\begin{aligned} &v_1 = (7a+b, -(2a+b), 9a+b), v_2 = (8a+b, -(10a+b), 11a+b), \\ &v_3 = (b, a+b, -(3a+b)), \text{ and } \quad v_4 = (4a+b, 5a+b, 6a+b), \end{aligned}$$

where a and b can be any real numbers. It is easy to calculate that

$$\begin{aligned} &P_{v_1}(0) + P_{v_2}(0) + P_{v_3}(0) + P_{v_4}(0) = 12b^2 + 132ab + 506a^2, \text{ and} \\ &P_{v_1}(t) + P_{v_2}(t) + P_{v_3}(t) + P_{v_4}(t) = 0, \text{ for } t = 1, 2. \end{aligned}$$

Then G_{61}, G_{63}, G_{65} can be obtained separately, as

$$G_{61} = \left(\begin{array}{rrr|rrr|rrr|rrr}
43 & -13 & 55 & 67 & -61 & 49 & -19 & 7 & 1 & 37 & 31 & 25 \\
55 & 43 & -13 & -61 & 49 & 67 & 7 & 1 & -19 & 31 & 25 & 37 \\
-13 & 55 & 43 & 49 & 67 & -61 & 1 & -19 & 7 & 25 & 37 & 31 \\
\hline
-67 & 61 & -49 & 43 & -13 & 55 & -31 & -37 & -25 & 7 & -19 & 1 \\
61 & -49 & -67 & 55 & 43 & -13 & -37 & -25 & -31 & -19 & 1 & 7 \\
-49 & -67 & 61 & -13 & 55 & 43 & -25 & -31 & -37 & 1 & 7 & -19 \\
\hline
19 & -7 & -1 & 31 & 37 & 25 & 43 & -13 & 55 & 61 & -67 & -49 \\
-7 & -1 & 19 & 37 & 25 & 31 & 55 & 43 & -13 & -67 & -49 & 61 \\
-1 & 19 & -7 & 25 & 31 & 37 & -13 & 55 & 43 & -49 & 61 & -67 \\
\hline
-37 & -31 & -25 & -7 & 19 & -1 & -61 & 67 & 49 & 43 & -13 & 55 \\
-31 & -25 & -37 & 19 & -1 & -7 & 67 & 49 & -61 & 55 & 43 & -13 \\
-25 & -37 & -31 & -1 & -7 & 19 & 49 & -61 & 67 & -13 & 55 & 43
\end{array}\right),$$

$$
G_{63} = \left(\begin{array}{rrr|rrr|rrr|rrr}
45 & -15 & 57 & 69 & -63 & 51 & -21 & 9 & 3 & 39 & 33 & 27 \\
57 & 45 & -15 & -63 & 51 & 69 & 9 & 3 & -21 & 33 & 27 & 39 \\
-15 & 57 & 45 & 51 & 69 & -63 & 3 & -21 & 9 & 27 & 39 & 33 \\
\hline
-69 & 63 & -51 & 45 & -15 & 57 & -33 & -39 & -27 & 9 & -21 & 3 \\
63 & -51 & -69 & 57 & 45 & -15 & -39 & -27 & -33 & -21 & 3 & 9 \\
-51 & -69 & 63 & -15 & 57 & 45 & -27 & -33 & -39 & 3 & 9 & -21 \\
\hline
21 & -9 & -3 & 33 & 39 & 27 & 45 & -15 & 57 & 63 & -69 & -51 \\
-9 & -3 & 21 & 39 & 27 & 33 & 57 & 45 & -15 & -69 & -51 & 63 \\
-3 & 21 & -9 & 27 & 33 & 39 & -15 & 57 & 45 & -51 & 63 & -69 \\
\hline
-39 & -33 & -27 & -9 & 21 & -3 & -63 & 69 & 51 & 45 & -15 & 57 \\
-33 & -27 & -39 & 21 & -3 & -9 & 69 & 51 & -63 & 57 & 45 & -15 \\
-27 & -39 & -33 & -3 & -9 & 21 & 51 & -63 & 69 & -15 & 57 & 45
\end{array}\right),
$$

$$
G_{65} = \left(\begin{array}{rrr|rrr|rrr|rrr}
47 & -17 & 59 & 71 & -65 & 53 & -23 & 11 & 5 & 41 & 35 & 29 \\
59 & 47 & -17 & -65 & 53 & 71 & 11 & 5 & -23 & 35 & 29 & 41 \\
-17 & 59 & 47 & 53 & 71 & -65 & 5 & -23 & 11 & 29 & 41 & 35 \\
\hline
-71 & 65 & -53 & 47 & -17 & 59 & -35 & -41 & -29 & 11 & -23 & 5 \\
65 & -53 & -71 & 59 & 47 & -17 & -41 & -29 & -35 & -23 & 5 & 11 \\
-53 & -71 & 65 & -17 & 59 & 47 & -29 & -35 & -41 & 5 & 11 & -23 \\
\hline
23 & -11 & -5 & 35 & 41 & 29 & 47 & -17 & 59 & 65 & -71 & -53 \\
-11 & -5 & 23 & 41 & 29 & 35 & 59 & 47 & -17 & -71 & -53 & 65 \\
-5 & 23 & -11 & 29 & 35 & 41 & -17 & 59 & 47 & -53 & 65 & -71 \\
\hline
-41 & -35 & -29 & -11 & 23 & -5 & -65 & 71 & 53 & 47 & -17 & 59 \\
-35 & -29 & -41 & 23 & -5 & -11 & 71 & 53 & -65 & 59 & 47 & -17 \\
-29 & -41 & -35 & -5 & -11 & 23 & 53 & -65 & 71 & -17 & 59 & 47
\end{array}\right).
$$

If $L_1 = (G_{61}^T, -G_{61}^T)^T, L_2 = (G_{63}^T, -G_{63}^T)^T$, and $L_3 = (G_{65}^T, -G_{65}^T)^T$, then $L = (L_1^T, L_2^T, L_3^T)^T$ is an $SOL(72, 12, 3)$, where each L_i is an orthogonal $L(24, 12)$.

Example 3. With $k = 2$, $p = 5$, an $SOL(80, 20, 2)$ can be obtained where each slice is an $L(40, 20)$. Take the vectors

$$
\begin{aligned}
v_1 &= (10a + b, 2a + b, -(13a + b), 14a + b, 11a + b), \\
v_2 &= (12a + b, 15a + b, 16a + b, 17a + b, -(18a + b)), \\
v_3 &= (19a + b, b, -(a + b), -(3a + b), -(4a + b)), \text{ and} \\
v_4 &= (5a + b, 6a + b, -(7a + b), 8a + b, -(9a + b)),
\end{aligned}
$$

where a and b can be any real numbers. It is easy to verify that

$$
\begin{aligned}
P_{v_1}(0) + P_{v_2}(0) + P_{v_3}(0) + P_{v_4}(0) &= 20b^2 + 380ab + 2470a^2, \text{ and} \\
P_{v_1}(t) + P_{v_2}(t) + P_{v_3}(t) + P_{v_4}(t) &= 0, \text{ for } t = 1, \ldots, 4.
\end{aligned}
$$

Then we can obtain G_{41} and G_{43} separately; they are presented in Appendix B. If $L_1 = (G_{41}^T, -G_{41}^T)^T$, and $L_2 = (G_{43}^T, -G_{43}^T)^T$, then $L = (L_1^T, L_2^T)^T$ is an $SOL(80, 20, 2)$, where each L_i is an orthogonal $L(40, 20)$.

2.3. Construction using Kharaghani arrays

The third construction method is based on an important result due to Kharaghani (2000).

Lemma 2 (Kharaghani (2000), Thm. 1). *Let $\{S_1, \ldots, S_8\}$ be a set of circulant matrices of order p, satisfying*

$$\sum_{i=1}^{8} S_i S_i^T = sI_p \text{ and } \sum_{i=1}^{4} (S_{2i-1}S_{2i}^T - S_{2i}S_{2i-1}^T) = 0.$$

Then the Kharaghani array

$$K(S_1, \ldots, S_8) = \begin{pmatrix} S_1 & S_2 & S_4R_p & S_3^TR_p & S_6R_p & S_5R_p & S_8R_p & S_7R_p \\ -S_2 & S_1 & S_3R_p & -S_4^TR_p & S_5R_p & -S_6R_p & S_7R_p & -S_8R_p \\ -S_4R_p & -S_3R_p & S_1 & S_2 & -S_8^TR_p & S_7^TR_p & S_6^TR_p & -S_5^TR_p \\ -S_3R_p & S_4R_p & -S_2 & S_1 & S_7^TR_p & S_8^TR_p & -S_5^TR_p & -S_6^TR_p \\ -S_6R_p & -S_5R_p & S_8^TR_p & -S_7^TR_p & S_1 & S_2 & -S_4^TR_p & S_3^TR_p \\ -S_5R_p & S_6R_p & -S_7^TR_p & -S_8^TR_p & -S_2 & S_1 & S_3^TR_p & S_4^TR_p \\ -S_8R_p & -S_7R_p & -S_6^TR_p & S_5^TR_p & S_4^TR_p & -S_3^TR_p & S_1 & S_2 \\ -S_7R_p & S_8R_p & S_5^TR_p & S_6^TR_p & -S_3^TR_p & -S_4^TR_p & -S_2 & S_1 \end{pmatrix}$$

is an orthogonal matrix of order $8p$.

Remark 1. As in Corollary 1, if there exist eight vectors of length p with zero PAF, they can be used to generate eight suitable circulant matrices for Lemma 2.

The key issue here is to obtain the sliced structure by constructing appropriate Kharaghani arrays.

Algorithm 3.

Step 1. Find eight vectors $v_1, \ldots, v_8$ of the same length p with zero PAF satisfying

$$\sum_{i=1}^{4} (C(v_{2i-1})C(v_{2i})^T - C(v_{2i})C(v_{2i-1})^T) = 0,$$

where the elements come from the set $\{b, b + a, \ldots, b + (8p - 1)a\}$ if the signs are ignored.

Step 2. Place the eight circulant matrices $C(v_1), \ldots, C(v_8)$ into a Kharaghani array to obtain $K_{ab} = K(C(v_1), \ldots, C(v_8))$.

Step 3. For a given k, set $L_i = (K_{2k,2i-1}^T, -K_{2k,2i-1}^T)^T$, for $i = 1, \ldots, k$. Obtain L by stacking $L_1, \ldots, L_k$ row by row, $L = (L_1^T, \ldots, L_k^T)^T$.

Theorem 3. *An L constructed by Algorithm 3 is an $SOL(16pk, 8p, k)$ with k slices $L_1, \ldots, L_k$, where L_i is an orthogonal $L(16p, 8p)$. When projected onto each dimension, each of the $16p$ equally-spaced intervals $[-16pk, -(16pk-2k)), \ldots, [-2k, 0), [0, 2k), \ldots, [16pk-2k, 16pk)$ contains exactly one point of each slice.*

Example 4. With $k = p = 3$, an $SOL(96, 24, 2)$ can be obtained where each slice is an $L(48, 24)$. Take the vectors

$$\begin{array}{ll} v_1 = (b, 13a+b, a+b), & v_2 = (11a+b, -(12a+b), 14a+b), \\ v_3 = (2a+b, 3a+b, -4a+b)), & v_4 = (15a+b, 16a+b, -(17a+b)), \\ v_5 = (5a+b, -(6a+b), -(7a+b)), & v_6 = (18a+b, 19a+b, 20a+b), \\ v_7 = (8a+b, 9a+b, -(10a+b)), \text{ and} & v_8 = (21a+b, 22a+b, -(23a+b)). \end{array}$$

It is easy to verify that

$$\sum_{i=0}^{8} P_{A_i}(0) = 24b^2 + 552ab + 4324a^2, \quad \sum_{i=0}^{8} P_{A_i}(t) = 0 \text{ for } t = 1, \ldots, 8, \text{ and}$$
$$\sum_{i=1}^{8} C(v_i)C(v_i)^T = cI_p, \text{ with } c = 24b^2 + 552ab + 4324a^2.$$

Then we have K_{41} and K_{43}; they can be found in Appendix B. If $L_1 = (K_{41}^T, -K_{41}^T)^T$ and $L_2 = (K_{43}^T, -K_{43}^T)^T$, then $L = (L_1^T, L_2^T)^T$ is an $SOL(96, 24, 2)$, where each slice is an orthogonal $L(48, 24)$.

3. Sliced Maximin-Orthogonal LHDs

In this section, the uniformity of LHDs measured by the maximin distance criterion is considered. Let $x_u = (x_u^1, \ldots, x_u^m)$ and $x_v = (x_v^1, \ldots, x_v^m)$ be two points in the experimental region $[-1, 1]^m$, the distance between them is defined as

$$d(x_u, x_v) = \left\{ \sum_{j=1}^{m} \left| x_u^j - x_v^j \right|^s \right\}^{1/s},$$

where different values of s correspond to different distances. Here, we take $s = 2$, Euclidean distance. Given a design D, the minimum inter-site distance of D is denoted by $d_{\min}(D)$, where

$$d_{\min}(D) = \min_{x_u, x_v \in D, u \neq v} d(x_u, x_v).$$

Due to Johnson, Moore, and Ylvisaker (1990), a maximin distance design D_* maximizes the minimum inter-site distance $d_{\min}(D)$,

$$d_{\min}(D_*) = \max_{D \in \mathcal{D}} d_{\min}(D),$$

where $\mathcal{D}$ is the set of possible designs. In this section, sliced maximin-orthogonal LHDs are searched by a column-exchange algorithm from the designs constructed in Section 2, where threshold accepting (see Dueck and Scheuer (1990)) is used as the accept rule.

Algorithm 4.

Step 1. Construct an initial $SOL(n,m,k)$, denoted by D_0, then scale the levels $\{-(n-1),-(n-3),\ldots,(n-1)\}$ to $\{-(n-1)/n,-(n-3)/n,\ldots,(n-1)/n\}$, and compute $d_{\min}(D_0)$, denoted by d_0. Set a sequence of threshold parameter $Th=(T_1,\ldots,T_L)$, where $T_1>\cdots>T_L>0$. Denote the iteration number by I under each T_l for $l=1,\ldots,L$. Set indices $l=1,\ i=1$.

Step 2. Hold the first slice of D_0 unchanged, randomly choose a slice from the remaining ones, and choose two columns in that slice; exchange these two columns in the slice and get a new sliced LHD, denoted by D_{try}. Compute $d_{\min}(D_{\text{try}})$, denoted by d_{try}.

Step 3. If $d_0-d_{\text{try}}\le T_l$, replace D_0 by D_{try} and set $d_0=d_{\text{try}}$; else leave D_0 unchanged.

Step 4. Update $i=i+1$, if $i\le I$, go to Step 2.

Step 5. Update $l=l+1$, if $l\le L$, reset $i=1$ and go to Step 2; else deliver $D_{\text{best}}=D_0$.

Remark 2. An optimal design can thus be obtained by exchanging the columns of some slices. It is obvious that each slice of the optimal sliced LHD is still orthogonal after the exchange. Thus the optimal sliced LHD remains orthogonal.

Example 5 (Example 1 continued)**.** Take the design L constructed in Example 1 as the initial design D_0, $D_0=L=(L_1^T,L_2^T)^T=(D_1^T,-D_1^T,D_3^T,-D_3^T)^T$, and scale the levels $\{-15,-13,\ldots,15\}$ to $\{-15/16,-13/16,\ldots,15/16\}$.

Compute $d_{\min}(D_0)$, and get $d_0=d_{\min}(D_0)=0.25$. Following Algorithm 4, we obtain the sliced maximin-orthogonal LHD D_{best} with $d_{\text{best}}=d_{\min}(D_{\text{best}})=0.9682$, which greatly improves the initial design D_0, where

$$D_{\text{best}}=\left(\begin{array}{rrrrrrrr|rrrrrrrr}1&5&9&13&-1&-5&-9&-13&3&7&11&15&-3&-7&-11&-15\\5&-1&13&-9&-5&1&-13&9&-15&-11&7&3&15&11&-7&-3\\-13&-9&5&1&13&9&-5&-1&11&-15&-3&7&-11&15&3&-7\\9&-13&-1&5&-9&13&1&-5&7&-3&15&-11&-7&3&-15&11\end{array}\right)^T.$$

In fact, the transformation from D_0 to D_{best} is that keeping the first slice L_1 of D_0 unchanged, and exchanging the four columns $1,2,3,4$ to $1,3,4,2$ in the second slice L_2. The improvement on $d_{\min}$ can also be viewed intuitively by the bivariate projections of D_0 and D_{best} in Figure 1, where X1, X2, X3 and X4 denote the four columns $1,2,3,4$ of D_0 and D_{best}, respectively.

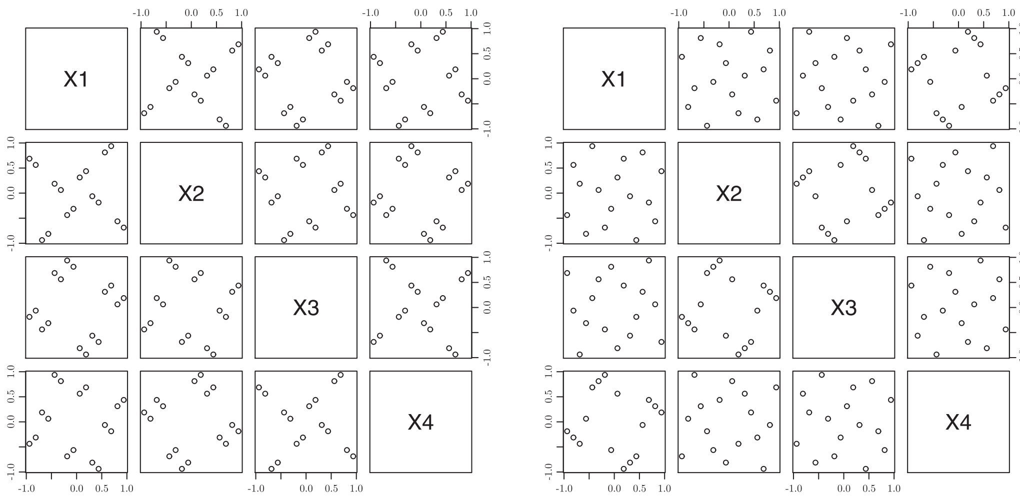

Figure 1. Bivariate projections among the four columns of D_0 and D_{best}.

Table 1. The $d_{\min}$'s of the initial sliced orthogonal LHDs and maximin-orthogonal ones.

$SOL(n,m,k)$	$(16,4,2)$	$(32,8,2)$	$(48,12,2)$	$(72,16,2)$	$(80,20,2)$	$(96,24,2)$	$(128,32,2)$
d_0	0.25	0.1768	0.1443	0.1250	0.2236	0.2041	0.0884
d_{best}	0.9682	1.5989	2.0613	2.4343	5.3193	5.9307	3.5029

Note: d_0 denotes the $d_{\min}$ of the initial design; d_{best} means the $d_{\min}$ of the optimal maximin-orthogonal one found by Algorithm 4.

Table 1 tabulates the $d_{\min}$'s of some optimal sliced maximin-orthogonal LHDs best found by Algorithm 4 and the corresponding initial designs, where the initial designs with 12 and 20 columns can be constructed by Algorithm 2, the one with 24 columns can be obtained by Algorithm 3, and those with $4, 8, 16$, and 32 columns can be easily obtained following Algorithm 1 and the construction of $OD(2^r)$ in Yang and Liu (2012). From Table 1, we can see that any of the sliced maximin-orthogonal LHDs gives a great improvement on the $d_{\min}$ compared with the corresponding initial one.

4. Further Results and Discussion

The newly constructed sliced maximin-orthogonal LHDs not only have the orthogonality but also a good uniformity (measured by the maximin distance). These are desirable properties for LHDs. The numbers of slices of the constructed orthogonal designs are rather flexible, not obtainable by all existing methods. Given an integer k, sliced LHDs with k slices can be constructed by Algorithms 1, 2, and 3. Each slice is an $L(2^{r+1}, 2^r)$ by Algorithm 1; an $L(8p, 4p)$ by Algorithm 2; and an $L(16p, 8p)$ by Algorithm 3. Then by Algorithm 4, the corresponding sliced

Table 2. The vectors for Goethals-Seidel arrays and Kharaghani arrays.

p	Order of array	Vectors
3	12	$v_1 = (7a+b, -(2a+b), 9a+b)$, $v_2 = (8a+b, -(10a+b), 11a+b)$, $v_3 = (b, a+b, -(3a+b))$, $v_4 = (4a+b, 5a+b, 6a+b)$.
3	24	$v_1 = (b, 13a+b, a+b)$, $v_2 = (11a+b, -(12a+b), 14a+b)$, $v_3 = (2a+b, 3a+b, -4a+b))$, $v_4 = (15a+b, 16a+b, -(17a+b))$, $v_5 = (5a+b, -(6a+b), -(7a+b))$, $v_6 = (18a+b, 19a+b, 20a+b)$, $v_7 = (8a+b, 9a+b, -(10a+b))$, $v_8 = (21a+b, 22a+b, -(23a+b))$.
5	20	$v_1 = (10a+b, 2a+b, -(13a+b), 14a+b, 11a+b)$, $v_2 = (12a+b, 15a+b, 16a+b, 17a+b, -(18a+b))$, $v_3 = (19a+b, b, -(a+b), -(3a+b), -(4a+b))$, $v_4 = (5a+b, 6a+b, -(7a+b), 8a+b, -(9a+b))$.

maximin-orthogonal LHDs can be obtained that are not available through other existing constructions. Combining fold-over structures and the sliced orthogonal LHDs constructed above, we have the following.

Theorem 4. *For any of the sliced orthogonal or maximin-orthogonal LHDs constructed by Algorithms* 1, ..., 4*, the full design and each slice have second-order orthogonality.*

Second-order orthogonality of a design is quite useful for effects screening and model fitting. In Step 1 of Algorithms 2 and 3, vectors with zero PAF should be found, and the algorithm in Section 3.1 of Georgiou and Efthimiou (2014) can be extended to search for such vectors. Details are omitted here. Table 2 lists the vectors for small p—they have been used in Examples 2, 3, and 4. With these vectors, sliced maximin-orthogonal LHDs with k slices can then be constructed for any $k \geq 2$.

Acknowledgements

The authors thank the reviewers for their constructive comments on an early version of this paper. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11271205, 11431006, 11101024, 11401321), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20130031110002), and the “131” Talents Program of Tianjin. The first two authors contributed equally to this work.

Appendix A: Proofs of Theorems

A.1. Proof of Theorem 1.

For $m = 2^r$, from the definitions of the OD and L_i, it is easy to verify that the levels of each column of L_i are

$$\{\pm(2i-1), \pm(2k+2i-1), \ldots, \pm(2k(m-1)+2i-1)\}, i = 1, \ldots, k.$$

Since L is obtained by stacking $L_1, \ldots, L_k$ row by row, the levels of each column of L are

$$\{\pm 1, \pm 3, \ldots, \pm(2km-1)\}.$$

Furthermore, the $2m$ levels of each L_i are in the equally-spaced intervals

$$[-2km, -2k(m-1)), [-2k(m-1), -2k(m-2)), \ldots, [-2k, 0), [0, 2k), \ldots, [2k(m-1), 2km).$$

This guarantees the sliced structure of L. The orthogonality of L and L_i follows from the orthogonality of the ODs.

A.2. Proof of Theorem 2.

We first show that L is a sliced LHD. From the structure of the Goethals-Seidel array G_{ab}, it is easy to see that each column of G_{ab} is a permutation of $\{b, b+a, \ldots, b+(4p-1)a\}$ without considering the signs. For a given k, from the definition of L_i, $i = 1, \ldots, k$, it is easy to verify that the levels of each column of L_i are $\{\pm(2i-1), \pm(2k+2i-1), \ldots, \pm(8pk-2k+2i-1)\}$, which correspond to the $8p$ equally-spaced intervals $[-8pk, -(8pk-2k)), \ldots, [-2k, 0), [0, 2k), \ldots, [8pk-2k, 8pk)$. This shows that L_i is an $L(8p, 4p)$. Furthermore, the elements in each column of L are $\{\pm 1, \pm 3, \ldots, \pm(8pk-1)\}$. This verifies that L is a sliced LHD with k sliced $L_1, \ldots, L_k$.

The orthogonality of L and L_i follows from the orthogonality of G_{ab} which comes from Corollary 1.

A.3. Proof of Theorem 3.

The proof of Theorem 3 is similar to that of Theorem 2.

Appendix B: Design Matrices for Examples 3 and 4

$$
G_{41}=\left(\begin{array}{rrrrr|rrrrr|rrrrr|rrrrr}
41 & 9 & -53 & 57 & 45 & -73 & 69 & 65 & 61 & 49 & -17 & -13 & -5 & 1 & 77 & -37 & 33 & -29 & 25 & 21 \\
45 & 41 & 9 & -53 & 57 & 69 & 65 & 61 & 49 & -73 & -13 & -5 & 1 & 77 & -17 & 33 & -29 & 25 & 21 & -37 \\
57 & 45 & 41 & 9 & -53 & 65 & 61 & 49 & -73 & 69 & -5 & 1 & 77 & -17 & -13 & -29 & 25 & 21 & -37 & 33 \\
-53 & 57 & 45 & 41 & 9 & 61 & 49 & -73 & 69 & 65 & 1 & 77 & -17 & -13 & -5 & 25 & 21 & -37 & 33 & -29 \\
9 & -53 & 57 & 45 & 41 & 49 & -73 & 69 & 65 & 61 & 77 & -17 & -13 & -5 & 1 & 21 & -37 & 33 & -29 & 25 \\
\hline
73 & -69 & -65 & -61 & -49 & 41 & 9 & -53 & 57 & 45 & -25 & 29 & -33 & 37 & -21 & 1 & -5 & -13 & -17 & 77 \\
-69 & -65 & -61 & -49 & 73 & 45 & 41 & 9 & -53 & 57 & 29 & -33 & 37 & -21 & -25 & -5 & -13 & -17 & 77 & 1 \\
-65 & -61 & -49 & 73 & -69 & 57 & 45 & 41 & 9 & -53 & -33 & 37 & -21 & -25 & 29 & -13 & -17 & 77 & 1 & -5 \\
-61 & -49 & 73 & -69 & -65 & -53 & 57 & 45 & 41 & 9 & 37 & -21 & -25 & 29 & -33 & -17 & 77 & 1 & -5 & -13 \\
-49 & 73 & -69 & -65 & -61 & 9 & -53 & 57 & 45 & 41 & -21 & -25 & 29 & -33 & 37 & 77 & 1 & -5 & -13 & -17 \\
\hline
17 & 13 & 5 & -1 & -77 & 25 & -29 & 33 & -37 & 21 & 41 & 9 & -53 & 57 & 45 & -61 & -65 & -69 & 73 & -49 \\
13 & 5 & -1 & -77 & 17 & -29 & 33 & -37 & 21 & 25 & 45 & 41 & 9 & -53 & 57 & -65 & -69 & 73 & -49 & -61 \\
5 & -1 & -77 & 17 & 13 & 33 & -37 & 21 & 25 & -29 & 57 & 45 & 41 & 9 & -53 & -69 & 73 & -49 & -61 & -65 \\
-1 & -77 & 17 & 13 & 5 & -37 & 21 & 25 & -29 & 33 & -53 & 57 & 45 & 41 & 9 & 73 & -49 & -61 & -65 & -69 \\
-77 & 17 & 13 & 5 & -1 & 21 & 25 & -29 & 33 & -37 & 9 & -53 & 57 & 45 & 41 & -49 & -61 & -65 & -69 & 73 \\
\hline
37 & -33 & 29 & -25 & -21 & -1 & 5 & 13 & 17 & -77 & 61 & 65 & 69 & -73 & 49 & 41 & 9 & -53 & 57 & 45 \\
-33 & 29 & -25 & -21 & 37 & 5 & 13 & 17 & -77 & -1 & 65 & 69 & -73 & 49 & 61 & 45 & 41 & 9 & -53 & 57 \\
29 & -25 & -21 & 37 & -33 & 13 & 17 & -77 & -1 & 5 & 69 & -73 & 49 & 61 & 65 & 57 & 45 & 41 & 9 & -53 \\
-25 & -21 & 37 & -33 & 29 & 17 & -77 & -1 & 5 & 13 & -73 & 49 & 61 & 65 & 69 & -53 & 57 & 45 & 41 & 9 \\
-21 & 37 & -33 & 29 & -25 & -77 & -1 & 5 & 13 & 17 & 49 & 61 & 65 & 69 & -73 & 9 & -53 & 57 & 45 & 41
\end{array}\right),
$$

$$
G_{43}=\left(\begin{array}{rrrrr|rrrrr|rrrrr|rrrrr}
43 & 11 & -55 & 59 & 47 & -75 & 71 & 67 & 63 & 51 & -19 & -15 & -7 & 3 & 79 & -39 & 35 & -31 & 27 & 23 \\
47 & 43 & 11 & -55 & 59 & 71 & 67 & 63 & 51 & -75 & -15 & -7 & 3 & 79 & -19 & 35 & -31 & 27 & 23 & -39 \\
59 & 47 & 43 & 11 & -55 & 67 & 63 & 51 & -75 & 71 & -7 & 3 & 79 & -19 & -15 & -31 & 27 & 23 & -39 & 35 \\
-55 & 59 & 47 & 43 & 11 & 63 & 51 & -75 & 71 & 67 & 3 & 79 & -19 & -15 & -7 & 27 & 23 & -39 & 35 & -31 \\
11 & -55 & 59 & 47 & 43 & 51 & -75 & 71 & 67 & 63 & 79 & -19 & -15 & -7 & 3 & 23 & -39 & 35 & -31 & 27 \\
\hline
75 & -71 & -67 & -63 & -51 & 43 & 11 & -55 & 59 & 47 & -27 & 31 & -35 & 39 & -23 & 3 & -7 & -15 & -19 & 79 \\
-71 & -67 & -63 & -51 & 75 & 47 & 43 & 11 & -55 & 59 & 31 & -35 & 39 & -23 & -27 & -7 & -15 & -19 & 79 & 3 \\
-67 & -63 & -51 & 75 & -71 & 59 & 47 & 43 & 11 & -55 & -35 & 39 & -23 & -27 & 31 & -15 & -19 & 79 & 3 & -7 \\
-63 & -51 & 75 & -71 & -67 & -55 & 59 & 47 & 43 & 11 & 39 & -23 & -27 & 31 & -35 & -19 & 79 & 3 & -7 & -15 \\
-51 & 75 & -71 & -67 & -63 & 11 & -55 & 59 & 47 & 43 & -23 & -27 & 31 & -35 & 39 & 79 & 3 & -7 & -15 & -19 \\
\hline
19 & 15 & 7 & -3 & -79 & 27 & -31 & 35 & -39 & 23 & 43 & 11 & -55 & 59 & 47 & -63 & -67 & -71 & 75 & -51 \\
15 & 7 & -3 & -79 & 19 & -31 & 35 & -39 & 23 & 27 & 47 & 43 & 11 & -55 & 59 & -67 & -71 & 75 & -51 & -63 \\
7 & -3 & -79 & 19 & 15 & 35 & -39 & 23 & 27 & -31 & 59 & 47 & 43 & 11 & -55 & -71 & 75 & -51 & -63 & -67 \\
-3 & -79 & 19 & 15 & 7 & -39 & 23 & 27 & -31 & 35 & -55 & 59 & 47 & 43 & 11 & 75 & -51 & -63 & -67 & -71 \\
-79 & 19 & 15 & 7 & -3 & 23 & 27 & -31 & 35 & -39 & 11 & -55 & 59 & 47 & 43 & -51 & -63 & -67 & -71 & 75 \\
\hline
39 & -35 & 31 & -27 & -23 & -3 & 7 & 15 & 19 & -79 & 63 & 67 & 71 & -75 & 51 & 43 & 11 & -55 & 59 & 47 \\
-35 & 31 & -27 & -23 & 39 & 7 & 15 & 19 & -79 & -3 & 67 & 71 & -75 & 51 & 63 & 47 & 43 & 11 & -55 & 59 \\
31 & -27 & -23 & 39 & -35 & 15 & 19 & -79 & -3 & 7 & 71 & -75 & 51 & 63 & 67 & 59 & 47 & 43 & 11 & -55 \\
-27 & -23 & 39 & -35 & 31 & 19 & -79 & -3 & 7 & 15 & -75 & 51 & 63 & 67 & 71 & -55 & 59 & 47 & 43 & 11 \\
-23 & 39 & -35 & 31 & -27 & -79 & -3 & 7 & 15 & 19 & 51 & 63 & 67 & 71 & -75 & 11 & -55 & 59 & 47 & 43
\end{array}\right).
$$

$K_{41}=$

$$
\left(\begin{array}{rrr|rrr|rrr|rrr|rrr|rrr|rrr|rrr}
1 & 53 & 5 & 45 & -49 & 57 & -69 & 65 & 61 & -17 & 13 & 9 & 81 & 77 & 73 & -29 & -25 & 21 & -93 & 89 & 85 & -41 & 37 & 33 \\
5 & 1 & 53 & 57 & 45 & -49 & 65 & 61 & -69 & 13 & 9 & -17 & 77 & 73 & 81 & -25 & 21 & -29 & 89 & 85 & -93 & 37 & 33 & -41 \\
53 & 5 & 1 & -49 & 57 & 45 & 61 & -69 & 65 & 9 & -17 & 13 & 73 & 81 & 77 & 21 & -29 & -25 & 85 & -93 & 89 & 33 & -41 & 37 \\
\hline
-45 & 49 & -57 & 1 & 53 & 5 & -17 & 13 & 9 & 69 & -65 & -61 & -29 & -25 & 21 & -81 & -77 & -73 & -41 & 37 & 33 & 93 & -89 & -85 \\
-57 & -45 & 49 & 5 & 1 & 53 & 13 & 9 & -17 & -65 & -61 & 69 & -25 & 21 & -29 & -77 & -73 & -81 & 37 & 33 & -41 & -89 & -85 & 93 \\
49 & -57 & -45 & 53 & 5 & 1 & 9 & -17 & 13 & -61 & 69 & -65 & 21 & -29 & -25 & -73 & -81 & -77 & 33 & -41 & 37 & -85 & 93 & -89 \\
\hline
69 & -65 & -61 & 17 & -13 & -9 & 1 & 53 & 5 & 45 & -49 & 57 & -89 & 93 & -85 & 37 & -41 & 33 & 77 & 81 & 73 & 25 & 29 & -21 \\
-65 & -61 & 69 & -13 & -9 & 17 & 5 & 1 & 53 & 57 & 45 & -49 & 93 & -85 & -89 & -41 & 33 & 37 & 81 & 73 & 77 & 29 & -21 & 25 \\
-61 & 69 & -65 & -9 & 17 & -13 & 53 & 5 & 1 & -49 & 57 & 45 & -85 & -89 & 93 & 33 & 37 & -41 & 73 & 77 & 81 & -21 & 25 & 29 \\
\hline
17 & -13 & -9 & -69 & 65 & 61 & -45 & 49 & -57 & 1 & 53 & 5 & 37 & -41 & 33 & 89 & -93 & 85 & 25 & 29 & -21 & -77 & -81 & -73 \\
-13 & -9 & 17 & 65 & 61 & -69 & -57 & -45 & 49 & 5 & 1 & 53 & -41 & 33 & 37 & -93 & 85 & 89 & 29 & -21 & 25 & -81 & -73 & -77 \\
-9 & 17 & -13 & 61 & -69 & 65 & 49 & -57 & -45 & 53 & 5 & 1 & 33 & 37 & -41 & 85 & 89 & -93 & -21 & 25 & 29 & -73 & -77 & -81 \\
\hline
-81 & -77 & -73 & 29 & 25 & -21 & 89 & -93 & 85 & -37 & 41 & -33 & 1 & 53 & 5 & 45 & -49 & 57 & -65 & 69 & -61 & 13 & -17 & 9 \\
-77 & -73 & -81 & 25 & -21 & 29 & -93 & 85 & 89 & 41 & -33 & -37 & 5 & 1 & 53 & 57 & 45 & -49 & 69 & -61 & -65 & -17 & 9 & 13 \\
-73 & -81 & -77 & -21 & 29 & 25 & 85 & 89 & -93 & -33 & -37 & 41 & 53 & 5 & 1 & -49 & 57 & 45 & -61 & -65 & 69 & 9 & 13 & -17 \\
\hline
29 & 25 & -21 & 81 & 77 & 73 & -37 & 41 & -33 & -89 & 93 & -85 & -45 & 49 & -57 & 1 & 53 & 5 & 13 & -17 & 9 & 65 & -69 & 61 \\
25 & -21 & 29 & 77 & 73 & 81 & 41 & -33 & -37 & 93 & -85 & -89 & -57 & -45 & 49 & 5 & 1 & 53 & -17 & 9 & 13 & -69 & 61 & 65 \\
-21 & 29 & 25 & 73 & 81 & 77 & -33 & -37 & 41 & -85 & -89 & 93 & 49 & -57 & -45 & 53 & 5 & 1 & 9 & 13 & -17 & 61 & 65 & -69 \\
\hline
93 & -89 & -85 & 41 & -37 & -33 & -77 & -81 & -73 & -25 & -29 & 21 & 65 & -69 & 61 & -13 & 17 & -9 & 1 & 53 & 5 & 45 & -49 & 57 \\
-89 & -85 & 93 & -37 & -33 & 41 & -81 & -73 & -77 & -29 & 21 & -25 & -69 & 61 & 65 & 17 & -9 & -13 & 5 & 1 & 53 & 57 & 45 & -49 \\
-85 & 93 & -89 & -33 & 41 & -37 & -73 & -77 & -81 & 21 & -25 & -29 & 61 & 65 & -69 & -9 & -13 & 17 & 53 & 5 & 1 & -49 & 57 & 45 \\
\hline
41 & -37 & -33 & -93 & 89 & 85 & -25 & -29 & 21 & 77 & 81 & 73 & -13 & 17 & -9 & -65 & 69 & -61 & -45 & 49 & -57 & 1 & 53 & 5 \\
-37 & -33 & 41 & 89 & 85 & -93 & -29 & 21 & -25 & 81 & 73 & 77 & 17 & -9 & -13 & 69 & -61 & -65 & -57 & -45 & 49 & 5 & 1 & 53 \\
-33 & 41 & -37 & 85 & -93 & 89 & 21 & -25 & -29 & 73 & 77 & 81 & -9 & -13 & 17 & -61 & -65 & 69 & 49 & -57 & -45 & 53 & 5 & 1
\end{array}\right),
$$

$K_{43} =$

$$\left(\begin{array}{rrr|rrr|rrr|rrr|rrr|rrr|rrr|rrr}
3 & 55 & 7 & 47 & -51 & 59 & -71 & 67 & 63 & -19 & 15 & 11 & 83 & 79 & 75 & -31 & -27 & 23 & -95 & 91 & 87 & -43 & 39 & 35 \\
7 & 3 & 55 & 59 & 47 & -51 & 67 & 63 & -71 & 15 & 11 & -19 & 79 & 75 & 83 & -27 & 23 & -31 & 91 & 87 & -95 & 39 & 35 & -43 \\
55 & 7 & 3 & -51 & 59 & 47 & 63 & -71 & 67 & 11 & -19 & 15 & 75 & 83 & 79 & 23 & -31 & -27 & 87 & -95 & 91 & 35 & -43 & 39 \\
\hline
-47 & 51 & -59 & 3 & 55 & 7 & -19 & 15 & 11 & 71 & -67 & -63 & -31 & -27 & 23 & -83 & -79 & -75 & -43 & 39 & 35 & 95 & -91 & -87 \\
-59 & -47 & 51 & 7 & 3 & 55 & 15 & 11 & -19 & -67 & -63 & 71 & -27 & 23 & -31 & -79 & -75 & -83 & 39 & 35 & -43 & -91 & -87 & 95 \\
51 & -59 & -47 & 55 & 7 & 3 & 11 & -19 & 15 & -63 & 71 & -67 & 23 & -31 & -27 & -75 & -83 & -79 & 35 & -43 & 39 & -87 & 95 & -91 \\
\hline
71 & -67 & -63 & 19 & -15 & -11 & 3 & 55 & 7 & 47 & -51 & 59 & -91 & 95 & -87 & 39 & -43 & 35 & 79 & 83 & 75 & 27 & 31 & -23 \\
-67 & -63 & 71 & -15 & -11 & 19 & 7 & 3 & 55 & 59 & 47 & -51 & 95 & -87 & -91 & -43 & 35 & 39 & 83 & 75 & 79 & 31 & -23 & 27 \\
-63 & 71 & -67 & -11 & 19 & -15 & 55 & 7 & 3 & -51 & 59 & 47 & -87 & -91 & 95 & 35 & 39 & -43 & 75 & 79 & 83 & -23 & 27 & 31 \\
\hline
19 & -15 & -11 & -71 & 67 & 63 & -47 & 51 & -59 & 3 & 55 & 7 & 39 & -43 & 35 & 91 & -95 & 87 & 27 & 31 & -23 & -79 & -83 & -75 \\
-15 & -11 & 19 & 67 & 63 & -71 & -59 & -47 & 51 & 7 & 3 & 55 & -43 & 35 & 39 & -95 & 87 & 91 & 31 & -23 & 27 & -83 & -75 & -79 \\
-11 & 19 & -15 & 63 & -71 & 67 & 51 & -59 & -47 & 55 & 7 & 3 & 35 & 39 & -43 & 87 & 91 & -95 & -23 & 27 & 31 & -75 & -79 & -83 \\
\hline
-83 & -79 & -75 & 31 & 27 & -23 & 91 & -95 & 87 & -39 & 43 & -35 & 3 & 55 & 7 & 47 & -51 & 59 & -67 & 71 & -63 & 15 & -19 & 11 \\
-79 & -75 & -83 & 27 & -23 & 31 & -95 & 87 & 91 & 43 & -35 & -39 & 7 & 3 & 55 & 59 & 47 & -51 & 71 & -63 & -67 & -19 & 11 & 15 \\
-75 & -83 & -79 & -23 & 31 & 27 & 87 & 91 & -95 & -35 & -39 & 43 & 55 & 7 & 3 & -51 & 59 & 47 & -63 & -67 & 71 & 11 & 15 & -19 \\
\hline
31 & 27 & -23 & 83 & 79 & 75 & -39 & 43 & -35 & -91 & 95 & -87 & -47 & 51 & -59 & 3 & 55 & 7 & 15 & -19 & 11 & 67 & -71 & 63 \\
27 & -23 & 31 & 79 & 75 & 83 & 43 & -35 & -39 & 95 & -87 & -91 & -59 & -47 & 51 & 7 & 3 & 55 & -19 & 11 & 15 & -71 & 63 & 67 \\
-23 & 31 & 27 & 75 & 83 & 79 & -35 & -39 & 43 & -87 & -91 & 95 & 51 & -59 & -47 & 55 & 7 & 3 & 11 & 15 & -19 & 63 & 67 & -71 \\
\hline
95 & -91 & -87 & 43 & -39 & -35 & -79 & -83 & -75 & -27 & -31 & 23 & 67 & -71 & 63 & -15 & 19 & -11 & 3 & 55 & 7 & 47 & -51 & 59 \\
-91 & -87 & 95 & -39 & -35 & 43 & -83 & -75 & -79 & -31 & 23 & -27 & -71 & 63 & 67 & 19 & -11 & -15 & 7 & 3 & 55 & 59 & 47 & -51 \\
-87 & 95 & -91 & -35 & 43 & -39 & -75 & -79 & -83 & 23 & -27 & -31 & 63 & 67 & -71 & -11 & -15 & 19 & 55 & 7 & 3 & -51 & 59 & 47 \\
\hline
43 & -39 & -35 & -95 & 91 & 87 & -27 & -31 & 23 & 79 & 83 & 75 & -15 & 19 & -11 & -67 & 71 & -63 & -47 & 51 & -59 & 3 & 55 & 7 \\
-39 & -35 & 43 & 91 & 87 & -95 & -31 & 23 & -27 & 83 & 75 & 79 & 19 & -11 & -15 & 71 & -63 & -67 & -59 & -47 & 51 & 7 & 3 & 55 \\
-35 & 43 & -39 & 87 & -95 & 91 & 23 & -27 & -31 & 75 & 79 & 83 & -11 & -15 & 19 & -63 & -67 & 71 & 51 & -59 & -47 & 55 & 7 & 3
\end{array}\right).$$

References

Ai, M. Y., He, Y. Z. and Liu S. M. (2012). Some new classes of orthogonal Latin hypercube designs. *J. Statist. Plann. Inference* **142**, 2809-2818.

Bingham, D., Sitter, R. R. and Tang, B. (2009). Orthogonal and nearly orthogonal designs for computer experiments. *Biometrika* **96**, 51-65.

Cioppa, T. M. and Lucas, T. M. (2007). Efficient nearly orthogonal and spaced-filling Latin hypercubes. *Technometrics* **49**, 45-55.

Dueck, G. and Scheuer, T. (1990). Threshold accepting: a general purpose optimization algorithm superior to simulated annealing. *J. Comput. Phys.* **90**, 161-175.

Georgiou, S. D. (2009). Orthogonal Latin hypercube designs from generalized orthogonal design. *J. Statist. Plann. Inference* **139**, 1530-1540.

Georgiou, S. D. and Efthimiou, E. (2014). Some classes of orthogonal Latin hypercube designs. *Statist. Sinica* **24**, 101-120.

Georgiou, S. D. and Stylianou, S. (2011). Block-circulant matrices for constructing optimal Latin hypercube designs. *J. Statist. Plann. Inference* **141**, 1933-1943.

Geramita, A. V. and Seberry, J. (1979). *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*. Marcel Dekker, New York.

Goethals, J. M. and Seidel, J. J. (1967). Orthogonal matrices with zero diagonal. *Canad. J. Math.* **19**, 1001-1010.

Huang, H. Z., Yang, J. F. and Liu, M. Q. (2013). Construction of sliced (nearly) orthogonal Latin hypercube designs. *J. Complexity* **30**, 355-365.

Johnson, M. E., Moore, L. M. and Ylvisaker, D. (1990). Minimax and maximin distance designs. *J. Statist. Plann. Inference* **26**, 131-148.

Kharaghani, H. (2000). Arrays for orthogonal designs. *J. Combin. Designs* **8**, 166-173.

Lin, C. D., Bingham, D., Sitter, R. R. and Tang, B. (2010). A new and flexible method for constructing designs for computer experiments. *Ann. Statist.* **38**, 1460-1477.

Lin, C. D., Mukerjee, R. and Tang, B. (2009). Construction of orthogonal and nearly orthogonal Latin hypercubes. *Biometrika* **96**, 243-247.

Morris, M. D. and Mitchell, T. J. (1995). Exploratory designs for computational experiments. *J. Statist. Plann. Inference* **43**, 381-402.

Pang, F., Liu, M. Q. and Lin, D. K. J. (2009). A construction method for orthogonal Latin hypercube designs with prime power levels. *Statist. Sinica* **19**, 1721-1728.

Qian, P. Z. G. (2012). Sliced Latin hypercube designs. *J. Amer. Statist. Assoc.* **107**, 393-399.

Steinberg, D. M. and Lin, D. K. J. (2006). A construction method for orthogonal Latin hypercube designs. *Biometrika* **93**, 279-288.

Sun, F. S., Liu, M. Q. and Lin, D. K. J. (2009). Construction of orthogonal Latin hypercube designs. *Biometrika* **96**, 971-974.

Sun, F. S., Liu, M. Q. and Lin, D. K. J. (2010). Construction of orthogonal Latin hypercube designs with flexible run sizes. *J. Statist. Plann. Inference* **140**, 3236-3242.

Yang, J. F., Lin, C. D., Qian, P. Z. G. and Lin, D. K. J. (2013). Construction of sliced orthogonal Latin hypercube designs. *Statist. Sinica* **23**, 1117-1130.

Yang, J. Y. and Liu, M. Q. (2012). Construction of orthogonal and nearly orthogonal Latin hypercube designs from orthogonal designs. *Statist. Sinica* **22**, 433-442.

Yang, X., Chen, H. and Liu, M. Q. (2014). Resolvable orthogonal array-based uniform sliced Latin hypercube designs. *Statist. Probab. Lett.* **93**, 108-115.

Ye, K. Q. (1998). Orthogonal column Latin hypercubes and their application in computer experiments. *J. Amer. Statist. Assoc.* **93**, 1430-1439.

Yin, Y. H., Lin, D. K. J. and Liu, M. Q. (2014). Sliced Latin hypercube designs via orthogonal arrays. *J. Statist. Plann. Inference* **149**, 162-171.

LPMC and Institute of Statistics, Nankai University, Tianjin 300071, China.

E-mail: jyyang@nankai.edu.cn

Department of Statistics, Tianjin University of Finance and Economics, Tianjin 300222, China.

E-mail: chlh1985@126.com

Department of Statistics, The Pennsylvania State University, University Park, PA 16802, USA.

E-mail: DKL5@psu.edu

LPMC and Institute of Statistics, Nankai University, Tianjin 300071, China.

E-mail: mqliu@nankai.edu.cn

(Received November 2013; accepted May 2015)