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A note on the construction of blocked two-level designs with general minimum lower order confounding

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ABSTRACT

Blocked designs are useful in experiments. The general minimum lower order confounding (GMC) is an elaborate criterion which was proposed for selecting optimal fractional factorial designs. Zhang and Mukerjee (2009b) extended the GMC criterion to the B-GMC criterion for selecting a $2^{n-m}:2^r$ design, where $2^{n-m}:2^r$ denotes a two-level regular blocked design with $N=2^{n-m}$ runs, n treatment factors and 2^r blocks. This paper gives the first construction method of B-GMC $2^{n-m}:2^r$ designs with $5N/16+1\le n\le N/2$. The results indicate that under isomorphism, with suitable choice of the blocking factors, each B-GMC blocked design has a common specific structure. Examples are included to illustrate the developed theory.

1. Introduction

Fractional factorial designs are widely used in scientific investigations and industrial experiments. When the experimental units are nonhomogeneous, blocking is an effective method for reducing systematic variation. Selecting a good blocked fractional factorial design is a subject of considerable important.

Zhang et al. (2008) introduced the aliased effect-number pattern (AENP) to characterize the two-level regular designs. Based on AENP, they proposed the general minimum lower order confounding (GMC) criterion for assessing the goodness of two-level designs. When the experimenter has some prior information concerning which factors may have much larger effects on the response, the GMC criterion can be used to select most appropriate designs. Zhang and Mukerjee (2009a) proposed the GMC criterion for s-level designs and derived explicit formulae connecting the key terms of the AENPs of the design and its complementary set. Under the GMC criterion, Cheng and Zhang (2010), Zhang and Cheng (2010), Chen and Liu (2011) and Li et al. (2011) completed the construction of the optimal 2^{n-m} designs with $n\ge N/4+1$, where $N=2^{n-m}$ being run size.

Zhang and Mukerjee (2009b) established a blocked GMC (B-GMC) criterion for selecting s-level regular blocked designs. They also developed a theory on constructing the optimal blocked designs under the B-GMC criterion and their results are

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especially useful when the number of two-level factors is larger than $N-16$, where N is the total number of runs. With a different consideration from that in Zhang and Mukerjee (2009b), Wei et al. (2014) proposed another blocked GMC (B^1-GMC) criterion for selecting optimal regular blocked designs. Zhao et al. (2013), Zhao and Sun (in press) and Zhao et al. (2016) completed the construction of all the B^1-GMC $2^{n-m}:2^r$ designs with $n \geq N/4+1$.

This paper studies the construction of B-GMC designs. This is clearly the first systematic construction method for B-GMC designs. Section 2 gives the definitions and notation. Section 3 develops the theory on constructing B-GMC $2^{n-m}:2^r$ designs with $5N/16+1 \leq n \leq N/2$. Some examples are provided. Conclusion is given in Section 4. All lengthy complicated proofs are provided in Appendix A and the supplementary material (see Appendix B).

2. Definitions and notation

Let $q=n-m$ and $N=2^q$. Define the $2^q \times (2^q-1)$ matrix

$$H_q = \{1, 2, 12, 3, 13, 23, 123, \ldots, 123\cdots q\}_{2^q}$$

with entries -1 and 1 and columns arranged in Yates order, where $1_{2^q} = (1,\ldots,1,\ -1,\ldots,-1)'$, $2_{2^q} = (1,\ldots,1,-1,\ldots,-1,1,\ldots,1,-1,\ldots,-1)'$, ..., and $q_{2^q} = (1,-1,\ldots,1,-1)'$ stand for its q independent columns, and the other columns are expressed as products of the q independent columns. Here we say q columns are independent if none of them can be expressed as products of other columns. The set of q independent columns may not be unique. Without confusion, we will omit the subscript 2^q hereafter. For example, consider $q=4$ and $H_4 = \{1, 2, 12, 3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234, 1234\}$. It can be verified that both $\{1, 2, 3, 4\}$ and $\{24, 34, 234, 124\}$ contain 4 independent columns.

Throughout this paper, we use $D=(D_t : D_b)$, taken from H_q, to denote a blocked $2^{n-m}:2^r$ design, where D_t is a $2^q \times n$ matrix with each column representing a treatment factor, and D_b is a $2^q \times (2^r-1)$ matrix for blocking. Of the n factors in D_t, $n-m$ are independent, and the remaining m are expressed as products of the $n-m$ independent factors. Of the 2^r-1 columns in D_b, r are considered to be independent block factors, which are also expressed as products of the $n-m$ independent treatment columns. The remaining columns in D_b are expressed as the products of the r independent columns, which are considered to be interactions of the r independent block factors.

To illustrate the above concepts, we consider a $2^{6-2}:2^2$ design $D=(D_t : D_b)$ with

$$\begin{aligned} D_t &= \{a_1 = 24, a_2 = 34, a_3 = 234, a_4 = 124, a_5 = 134, a_6 = 1234\}, \\ D_b &= \{b_1 = 1, b_2 = 4, b_3 = 14\}. \end{aligned} \tag{1}$$

Here $a_1,\ldots,a_6$ are the labels for 6 treatment factors and b_1, b_2, b_3 are the labels for 3 block factors. In this design, $a_1,\ldots,a_4$ are four independent treatment factors. The remaining two treatment factors are determined by $a_5 = a_1a_2a_4$ and $a_6 = a_1a_3a_4$, which imply

$$I = a_1a_2a_4a_5 = a_1a_3a_4a_6 = a_2a_3a_5a_6, \tag{2}$$

where I is a $2^q \times 1$ column with all elements being 1 and is called the identify element, and $a_2a_3a_5a_6$ is the product of $a_1a_2a_4a_5$ and $a_1a_3a_4a_6$. In (2), $a_1a_2a_4a_5$, $a_1a_3a_4a_6$, and $a_2a_3a_5a_6$ are called treatment-defining words, in which $a_1a_2a_4a_5$ and $a_1a_3a_4a_6$ are independent. Among the 3 block factors, the two independent block factors are determined by $b_1 = a_1a_4$ and $b_2 = a_1a_2a_3$ or equivalently

$$I = a_1a_4b_1 = a_1a_2a_3b_2,$$

in which $a_1a_4b_1$ and $a_1a_2a_3b_2$ are called independent block-defining words. The remaining block factor b_3 is determined by $b_3 = b_1b_2 = a_2a_3a_4$.

For a general $2^{n-m}:2^r$ design $D=(D_t : D_b)$, m treatment factors are expressed as the products of $n-m$ independent treatment columns, which determine m independent treatment-defining words; r independent block factors are expressed as the products of $n-m$ independent treatment columns, which determine r independent block-defining words. All possible products of the m treatment-defining words constitute an identical subgroup, called the treatment-defining contrast subgroup. Each element in this subgroup is called a treatment-defining word. The number of letters in a treatment-defining word is referred to its length. Let A_{i0} be the number of treatment-defining words of length i in D_t, and A_{i1} be the number of ith order treatment effects aliased with a block effect with $1 \leq i \leq n$. The design D_t is said to have resolution R if its shortest treatment-defining word has length R (Box and Hunter, 1961).

We now review the B-GMC criterion for blocked designs, introduced by Zhang and Mukerjee (2009b). For $1 \leq i \leq n$, let ${}_i^{\#}C_0(D)$ denote the number of ith order treatment effects which is neither the treatment-defining words nor aliased with the block effects. Then ${}_i^{\#}C_0(D) = K_i - A_{i0} - A_{i1}$, where $K_i = n!/\{i!(n-i)!\}$ is the total number of the ith order treatment effects. Further, among the ${}_i^{\#}C_0(D)$ effects which might be estimated, let ${}_i^{\#}C_j^{(k)}(D)$ denote the number of those aliased with k jth order effects, and let

$${}_i^{\#}C_j(D) = \left({}_i^{\#}C_j^{(0)}(D), {}_i^{\#}C_j^{(1)}(D), \ldots, {}_i^{\#}C_j^{(K_j)}(D)\right)$$

for $1 \leq i, j \leq n$.

The B-GMC criterion for blocked designs aims at sequentially maximizing the components of

$$^{\#}C(D) = \left({}^{\#}_{1}C_2(D), {}^{\#}_{2}C_0(D), {}^{\#}_{2}C_2(D), {}^{\#}_{1}C_3(D), {}^{\#}_{2}C_3(D), {}^{\#}_{3}C_0(D), {}^{\#}_{3}C_2(D), {}^{\#}_{3}C_3(D), \ldots\right).$$

This is called the aliased effect-number pattern of the blocked designs (B-AENP for short). When all the treatment effects involving three or more factors are negligible, the B-AENP $^{\#}C(D)$ is reduced to

$$^{\#}C(D) = \left\{{}^{\#}_{1}C_2(D), {}^{\#}_{2}C_0(D), {}^{\#}_{2}C_2(D)\right\}. \tag{3}$$

A design D is said to have B-GMC if it sequentially maximizes the components of (3).

We finish this section by defining more notation, which serves as the preparation for next section. We first define some submatrices of H_q. Let

$$H_1 = \{1\}, \qquad H_r = \{H_{r-1}, r, rH_{r-1}\}, \quad \text{for } r = 2, \ldots, q,$$

where $rH_{r-1} = \{rd : d \in H_{r-1}\}$ and H_r consists of the first $2^r - 1$ columns of H_q. Further, let $H_{l,l'}$ be the subset of H_q which consists of the vectors $l, \ldots, l'$ and all their possible products. For example, $H_{2,4} = \{2, 3, 23, 4, 24, 34, 234\}$. When $l = 1$, we simplify the notation $H_{l,l'}$ as $H_{l'}$. Moreover, let $F_{l,1} = \{l\}$ and $F_{l,l'} = \{l, lH_{l'-1}\}$ with $2 \le l' \le l$. For example, $F_{4,3} = \{4, 14, 24, 124\}$.

For a given set $Q \subset H_q$ and a column $d \in H_q$, define

$$B_i(Q, d) = \left|\left\{(d_1, \ldots, d_i) : d_1, \ldots, d_i \in Q, d_1 \cdots d_i = d\right\}\right|.$$

Hereafter, $|\cdot|$ denotes the cardinality of a set and $d_1 \cdots d_i$ means the interaction of $d_1, \ldots, d_i$. By the definition, $B_i(Q, d)$ is the number of ith order interactions of Q aliased with d. For example, consider $q = 4$, $Q = \{1, 2, 13, 23, 14, 24\} \subset H_4$, $i = 2$, and $d = 12 \in H_4$. Among the 15 two-factor interactions in Q, there are three two-factor interactions (between 1 and 2, between 13 and 23, and between 14 and 24), which are aliased with d. Hence, $B_2(Q, d) = 3$.

3. B-GMC $2^{n-m} : 2^r$ designs with $5N/16 + 1 \le n \le N/2$ factors

In the following, both D_t and D_b are treated as subsets of H_q. Two $2^{n-m} : 2^r$ designs $D^{(1)} = (D_t^{(1)} : D_b^{(1)})$ and $D^{(2)} = (D_t^{(2)} : D_b^{(2)})$ are said to be isomorphic (or equivalent) if there exists an isomorphism mapping that maps each column of $D_t^{(1)}$ to some column of $D_t^{(2)}$, and each column of $D_b^{(1)}$ to some column of $D_b^{(2)}$. Suppose that $D = (D_t : D_b)$ is a $2^{n-m} : 2^r$ design with $5N/16 + 1 \le n \le N/2$. Zhao et al. (2013) showed that if ${}^{\#}_{1}C_2(D)$ is maximized, then D_t must be taken from a saturated resolution 4 design with q independent columns. Up to isomorphism, suppose $D_t \subset F_{q,q}$. Then, there are two possible choices for D_b, as in Zhao et al. (2013).

Lemma 1 (*Zhao et al., 2013*)**.** *Suppose $D = (D_t : D_b)$ is a $2^{n-m} : 2^r$ design. If $D_t \subset F_{q,q}$, up to isomorphism, then there are two possibilities for the block effects D_b:* (i) $D_b = H_r$ *and* (ii) $D_b = H_{r-1} \cup F_{q,r}$.

Let $D^* = (D_t^* : D_b^*)$ with $D_t^* = F_{q,q} \backslash D_t$ and $D_b^* = H_{q-1} \cap D_b$. Lemma 2 describes the relation between the B-AENPs of D and D^*. The relationship is very helpful for constructing B-GMC designs.

Lemma 2. *Suppose $D = (D_t : D_b)$ is a $2^{n-m} : 2^r$ design with $D_t \subset F_{q,q}$, $q \ge 4$ and $n > N/4$. Let $D^* = (D_t^* : D_b^*)$ with $D_t^* = F_{q,q} \backslash D_t$ and $D_b^* = H_{q-1} \cap D_b$. Then we have*

(a) ${}^{\#}_{1}C_2^{(k)}(D) = \begin{cases} n & \text{if } k = 0, \\ 0 & \text{if } k \ge 1. \end{cases}$

(b) ${}^{\#}_{2}C_2^{(k)}(D) = \begin{cases} 0 & \text{if } k < n - N/4 - 1, \\ (k+1)(N/2 - 1 - f(D^*)) & \text{if } k = n - N/4 - 1, \\ (k+1)/(k+1-n+N/4)\, {}^{\#}_{2}C_2^{(k-n+N/4)}(D^*) & \text{if } k > n - N/4 - 1, \end{cases}$

where $f(D^*) = |D_b^*| + |\{d : d \in H_q \backslash D_b^*, B_2(D_t^*, d) > 0\}|$.

(c) ${}^{\#}_{2}C_0(D) = {}^{\#}_{2}C_0(D^*) + (n - N/4)\left|H_{q-1} \backslash D_b^*\right|$.

Lemma 2 implies that a design $D = (D_t : D_b)$ has B-GMC if and only if $D_t \subset F_{q,q}$ and the corresponding D^* maximizes

$$\left({}^{\#}_{2}C_0(D^*) + (n - N/4)\left|H_{q-1} \backslash D_b^*\right|, -f(D^*), {}^{\#}_{2}C_2(D^*)\right). \tag{4}$$

By Lemma 1, there are two classes of blocked designs $D = (D_t : D_b)$ with $D_t \subset F_{q,q}$. One has $D_b = H_r$ and the other has $D_b = H_{r-1} \cup F_{q,r}$. We will find the B-GMC designs in each class, and then compare the two B-GMC designs to give the B-GMC design among all the $2^{n-m} : 2^r$ designs. In each class, $(n - N/4)\left|H_{q-1} \backslash D_b^*\right|$ is a constant and (4) can be simplified to

$$\left({}^{\#}_{2}C_0(D^*), -f(D^*), {}^{\#}_{2}C_2(D^*)\right). \tag{5}$$

Then we have the following useful theorem for constructing B-GMC $2^{n-m} : 2^r$ designs with $5N/16 + 1 \le n \le N/2$.

Theorem 3. *Suppose that $D = (D_t : D_b)$ is a $2^{n-m} : 2^r$ design with $D_t \subset F_{q,q}$. Let $D^* = (D_t^* : D_b^*)$ with $D_t^* = F_{q,q} \backslash D_t$ and $D_b^* = H_{q-1} \cap D_b$.*

(a) *When $N/2 - 2^{r-1} + 1 \le n \le N/2$, D has B-GMC if and only if $D_b = H_r$ and $D^* = (D_t^* : D_b^*)$ maximizes (5);*
(b) *When $5N/16 + 1 \le n \le N/2 - 2^{r-1}$, D has B-GMC if and only if $D_b = H_{r-1} \cup F_{q,r}$ and $D^* = (D_t^* : D_b^*)$ maximizes (5).*

For ease of presentation, we need some new notation. Let $\alpha_0 = \beta_0 = I$, the column with all entries 1,

$$H_{r+1,q-1} = \{\alpha_1, \ldots, \alpha_{2^{q-r-1}-1}\}, \quad \text{and} \quad H_{r,q-1} = \{\beta_1, \ldots, \beta_{2^{q-r}-1}\}.$$

Further assume that the elements α_i and β_j are in Yates order, respectively. For example, $\alpha_1 = r + 1$, $\alpha_2 = r + 2$, $\alpha_3 = (r + 1)(r + 2)$, $\beta_1 = r$, $\beta_2 = r + 1$ and $\beta_3 = r(r + 1)$. Then, we can express $F_{q,q}$ as

$$F_{q,q} = \bigcup_{i=0}^{2^{q-r-1}-1} \alpha_i F_{q,r+1} = \bigcup_{j=0}^{2^{q-r}-1} \beta_j F_{q,r}.$$

Arrange the elements of $\alpha_i F_{q,r+1}$ and $\beta_j F_{q,r}$ in Yates order, we have the following theorem.

Theorem 4. *Suppose that $D = (D_t : D_b)$ is a $2^{n-m} : 2^r$ design.*

(a) *When $N/2 - 2^{r-1} + 1 \le n \le N/2$, if D_t consists of the last $n_1 - 1$ columns of $\alpha_i F_{q,r+1}$ for $i = 0, 1, \ldots, J_1 - 1$, the last n_1 columns of $\alpha_i F_{q,r+1}$ for $i = J_1, \ldots, 2^{q-r-1} - 1$, and $D_b = H_r$, then D has B-GMC. Here, $n_1 = \left\lceil \frac{n}{2^{q-r-1}} \right\rceil$, $J_1 = 2^{q-r-1} n_1 - n$ and $\lceil x \rceil$ denotes the smallest integer which is larger than or equal to x.*
(b) *When $5N/16 + 1 \le n \le N/2 - 2^{r-1}$ and $\left\lceil \frac{N/2 - 2^{r-1} - n}{2^{q-r}-1} \right\rceil$ is odd (or even), if D_t consists of the last n_2 (or $n_2 - 1$) columns of $\beta_j F_{q,r}$ for $j = 1, \ldots, J_2$, the last $n_2 - 1$ (or n_2) columns of $\beta_j F_{q,r}$ for $j = J_2 + 1, \ldots, 2^{q-r} - 1$, and $D_b = H_{r-1} \cup F_{q,r}$, then D has B-GMC. Here $n_2 = \left\lceil \frac{n}{2^{q-r}-1} \right\rceil$ and $J_2 = n - (2^{q-r} - 1)(n_2 - 1)$ (or $J_2 = (2^{q-r} - 1)n_2 - n$).*

The next examples show the usefulness of Theorem 4 for the construction of B-GMC designs.

Example 1. We first consider the construction of B-GMC $2^{6-2} : 2^2$ design. Here $n = 6, m = 2, r = 2, q = 4$, and $N = 16$. Then $5N/16 + 1 \le n \le N/2 - 2^{r-1}$ and $\left\lceil \frac{N/2 - 2^{r-1} - n}{2^{q-r}-1} \right\rceil = \left\lceil \frac{8-2-6}{2^{4-2}-1} \right\rceil = 0$ is even. Recall that $\beta_0 = I$, $H_{r,q-1} = H_{2,3} = \{\beta_1, \ldots, \beta_3\} = \{2, 3, 23\}$, and $F_{q,r} = F_{4,2} = \{4, 14\}$. Then

$$F_{q,q} = F_{4,4} = \bigcup_{j=0}^{3} \beta_j F_{q,r} = \left\{ \begin{matrix} 4 & 14 \\ 24 & 124 \\ 34 & 134 \\ 234 & 1234 \end{matrix} \right\}.$$

Note that $n_2 = \left\lceil \frac{n}{2^{q-r}-1} \right\rceil = \left\lceil \frac{6}{2^{4-2}-1} \right\rceil = 2$ and $J_2 = (2^{q-r} - 1)n_2 - n = 0$. By (b) of Theorem 4, $D = (D_t : D_b)$ with

$$D_t = \left\{ \begin{matrix} 24 & 124 \\ 34 & 134 \\ 234 & 1234 \end{matrix} \right\}$$

and $D_b = H_{r-1} \cup F_{q,r} = \{1, 4, 14\}$, which is exactly the design in (1), has B-GMC among all $2^{6-2} : 2^2$ designs.

Suppose that we still use b_1 and b_2 to label the first two block factors 1 and 4, as we have done in (1). After rearranging its rows, the B-GMC $2^{6-2} : 2^2$ design is presented in Table 1, in which the 16 runs are divided into four blocks I, II, III, IV, according to $(b_1, b_2) = (-1, -1), (-1, 1), (1, -1)$, and $(1, 1)$.

Example 2. We next consider the construction of B-GMC $2^{n-m} : 2^r$ designs with $q = 6$, $N = 64$, $r = 3$, and $5N/16 + 1 \le n \le N/2$.

If we take $n = 29$, then $N/2 - 2^{r-1} + 1 \le n \le N/2$. We have

$$H_r = H_3 = \{1, 2, 12, 3, 13, 23, 123\} \quad \text{and} \quad H_{r+1,q-1} = H_{4,5} = \{\alpha_1, \alpha_2, \alpha_3\} = \{4, 5, 45\}.$$

Note that $\alpha_0 = I$ and $F_{q,r+1} = F_{6,4} = \{6, 16, 26, 126, 36, 136, 236, 1236\}$. Then

$$F_{6,6} = \bigcup_{i=0}^{3} \alpha_i F_{6,4} = \left\{ \begin{matrix} 6 & 16 & 26 & 126 & 36 & 136 & 236 & 1236 \\ 46 & 146 & 246 & 1246 & 346 & 1346 & 2346 & 12346 \\ 56 & 156 & 256 & 1256 & 356 & 1356 & 2356 & 12356 \\ 456 & 1456 & 2456 & 12456 & 3456 & 13456 & 23456 & 123456 \end{matrix} \right\}.$$

Table 1
The B-GMC $2^{6-2}:2^2$ design $D=(D_t:D_b)$ with $D_t=\{a_1=24, a_2=34, a_3=234, a_4=124, a_5=134, a_6=1234\}$ and $D_b=\{b_1=1, b_2=4, b_3=14\}$.

Treatment factors						Block factors			Blocks
a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	
−1	−1	−1	−1	−1	−1	1	−1	−1	III
−1	−1	−1	1	1	1	−1	−1	1	I
−1	−1	1	−1	−1	1	1	1	1	IV
−1	−1	1	1	1	−1	−1	1	−1	II
−1	1	−1	−1	1	−1	1	1	1	IV
−1	1	−1	1	−1	1	−1	1	−1	II
−1	1	1	−1	1	1	1	−1	−1	III
−1	1	1	1	−1	−1	−1	−1	1	I
1	−1	−1	−1	1	1	−1	1	−1	II
1	−1	−1	1	−1	−1	1	1	1	IV
1	−1	1	−1	1	−1	−1	−1	1	I
1	−1	1	1	−1	1	1	−1	−1	III
1	1	−1	−1	−1	1	−1	−1	1	I
1	1	−1	1	1	−1	1	−1	−1	III
1	1	1	−1	−1	−1	−1	1	−1	II
1	1	1	1	1	1	1	1	1	IV

Further, $n_1=\left\lceil\frac{n}{2^{q-r-1}}\right\rceil=\left\lceil\frac{29}{2^{6-3-1}}\right\rceil=8$ and $J_1=2^{q-r-1}n_1-n=4\times 8-29=3$. By (a) of Theorem 4, $D_1=(D_{t1}:D_{b1})$ with

$$D_{t1}=\left\{\begin{array}{rrrrrrrr} & 16 & 26 & 126 & 36 & 136 & 236 & 1236\\ & 146 & 246 & 1246 & 346 & 1346 & 2346 & 12346\\ & 156 & 256 & 1256 & 356 & 1356 & 2356 & 12356\\ 456 & 1456 & 2456 & 12456 & 3456 & 13456 & 23456 & 123456\end{array}\right\}$$

and $D_{b1}=H_3$ has B-GMC among the $2^{29-23}:2^3$ designs.

If we take $n=23$, then $5N/16+1\le n\le N/2-2^{r-1}$ and $\left\lceil\frac{N/2-2^{r-1}-n}{2^{q-r}-1}\right\rceil=\left\lceil\frac{32-4-23}{2^{5-3}-1}\right\rceil=1$ is odd. Recall that

$$H_{r-1}=H_2=\{1,2,12\}\quad\text{and}\quad H_{r,q-1}=H_{3,5}=\{\beta_1,\ldots,\beta_7\}=\{3,4,34,5,35,45,345\},$$

and note that $\beta_0=I$ and $F_{q,r}=F_{6,3}=\{6,16,26,126\}$. Then

$$F_{6,6}=\bigcup_{j=0}^{7}\beta_j F_{6,3}=\left\{\begin{array}{rrrr} 6 & 16 & 26 & 126\\ 36 & 136 & 236 & 1236\\ 46 & 146 & 246 & 1246\\ 346 & 1346 & 2346 & 12346\\ 56 & 156 & 256 & 1256\\ 356 & 1356 & 2356 & 12356\\ 456 & 1456 & 2456 & 12456\\ 3456 & 13456 & 23456 & 123456\end{array}\right\}.$$

Note that $n_2=\left\lceil\frac{n}{2^{q-r}-1}\right\rceil=\left\lceil\frac{23}{2^{6-3}-1}\right\rceil=4$ and $J_2=n-(2^{q-r}-1)(n_2-1)=2$. By (b) of Theorem 4, $D_2=(D_{t2}:D_{b2})$ with

$$D_{t2}=\left\{\begin{array}{rrrr} 36 & 136 & 236 & 1236\\ 46 & 146 & 246 & 1246\\ & 1346 & 2346 & 12346\\ & 156 & 256 & 1256\\ & 1356 & 2356 & 12356\\ & 1456 & 2456 & 12456\\ & 13456 & 23456 & 123456\end{array}\right\}$$

and $D_{b2}=H_2\cup F_{6,3}=\{1,2,12,6,16,26,126\}$ has B-GMC among the $2^{23-17}:2^3$ designs.

4. Conclusion

B-GMC criterion proposed by Zhang and Mukerjee (2009b) is an improvement over the conventional GMC and minimum aberration criteria. Our paper is clearly the first study on the systematic construction method of the B-GMC designs. Of course, Zhang and Mukerjee (2009b) also construct few B-GMC design as examples. Here, we provide all B-GMC designs with all n between $5N/16+1$ and $N/2$, although other n are also available upon request.

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Appendix A

A.1. Proof of Lemma 2

Parts (a) and (b) are results of Lemma 2 in Zhao et al. (2013). We need only to prove (c). According to the definition of ${}_2^{\#}C_0(D)$, we have

$$ {}_2^{\#}C_0(D) = \sum_{k>0} k \cdot \left|\{d : d \in H_q \backslash D_b, B_2(D_t, d) = k\}\right|. $$

Zhao et al. (2013) showed that $B_2(D_t, d) = B_2(D_t^*, d) + n - N/4$ if $d \in H_{q-1}$ and 0 otherwise. Combining the fact that $d_1 d_2 \in H_{q-1}$ for any pair (d_1, d_2) in $D_t (\subset F_{q,q})$, we get

$$ \begin{aligned} {}_2^{\#}C_0(D) &= \sum_{k>0} k \cdot \left|\{d : d \in H_{q-1} \backslash D_b^*, B_2(D_t^*, d) = k - n + N/4\}\right| \\ &= \sum_{k \geq n-N/4} k \cdot \left|\{d : d \in H_{q-1} \backslash D_b^*, B_2(D_t^*, d) = k - n + N/4\}\right|. \end{aligned} $$

Changing $k - n + N/4$ to h, we obtain

$$ \begin{aligned} {}_2^{\#}C_0(D) &= \sum_{h \geq 0} (h + n - N/4) \left|\{d : d \in H_{q-1} \backslash D_b^*, B_2(D_t^*, d) = h\}\right| \\ &= \sum_{h>0} h \cdot \left|\{d : d \in H_{q-1} \backslash D_b^*, B_2(D_t^*, d) = h\}\right| + (n - N/4) \left|H_{q-1} \backslash D_b^*\right|. \end{aligned} $$

By the fact that $B_2(D_t^*, d) = 0$ for any $d \in F_{q,q}$, we have

$$ \begin{aligned} {}_2^{\#}C_0(D) &= \sum_{h>0} h \cdot \left|\{d : d \in H_q \backslash D_b^*, B_2(D_t^*, d) = h\}\right| + (n - N/4) \left|H_{q-1} \backslash D_b^*\right| \\ &= {}_2^{\#}C_0(D^*) + (n - N/4) \left|H_{q-1} \backslash D_b^*\right|. \end{aligned} $$

This completes the proof. □

A.2. Proof of Theorem 3

For (a). By Lemma 1, there are two classes of blocked designs. One class contains designs with $D_b = H_r$ and the other one contains those with $D_b = H_{r-1} \cup F_{q,r}$. Note that when $N/2 - 2^{r-1} + 1 \leq n \leq N/2$, the number of columns of $F_{q,q} \backslash F_{q,r}$, which is $N/2 - 2^{r-1}$, is smaller than that of D_t, the second class does not exist, we only need to consider the case that $D_b = H_r$. Then (a) follows directly.

For (b). Suppose that $\tilde{D} = (\tilde{D}_t : D_b)$ has B-GMC among the designs $D = (D_t : D_b)$ with $D_t \subset F_{q,q}$ and $D_b = H_{r-1} \cup F_{q,r}$, and $\tilde{E} = (\tilde{E}_t : E_b)$ has B-GMC among the designs $E = (E_t : E_b)$ with $E_t \subset F_{q,q}$ and $E_b = H_r$. To prove Part (b), it suffices to show that ${}_2^{\#}C_0(\tilde{D}) > {}_2^{\#}C_0(\tilde{E})$. By the definition of ${}_2^{\#}C_0$, we have

$$ {}_2^{\#}C_0(\tilde{D}) = n(n-1)/2 - A_{21}(\tilde{D}) \quad \text{and} \quad {}_2^{\#}C_0(\tilde{E}) = n(n-1)/2 - A_{21}(\tilde{E}). $$

So, we need only to show $A_{21}(\tilde{D}) < A_{21}(\tilde{E})$.

From the definition of A_{21}, $A_{21}(D)$ is the number of pairs (d_1, d_2) in D_t such that $d_1 d_2 \in D_b$. When $D_t \subset F_{q,q}$ and $D_b = H_{r-1} \cup F_{q,r}$, we have $D_t \cap F_{q,r} = \emptyset$, the empty set, and hence

$$ D_t \subset F_{q,q} \backslash F_{q,r} = \bigcup_{j=1}^{2^{q-r}-1} \beta_j F_{q,r}. $$

Let $D_{tj} = D_t \cap (\beta_j F_{q,r})$ for $j = 1, \ldots, 2^{q-r} - 1$. For a pair (d_1, d_2) of D_t, $d_1 d_2 \in D_b$ if and only if both d_1 and d_2 are in the same D_{tj} for some j. Thus, we have

$$A_{21}(D) = \sum_{j=1}^{2^{q-r}-1} |D_{tj}|(|D_{tj}| - 1)/2 = \sum_{j=1}^{2^{q-r}-1} |D_{tj}|^2/2 - n/2.$$

Note that $E_t \subset F_{q,q} = \bigcup_{i=0}^{2^{q-r-1}-1} \alpha_i F_{q,r+1}$. Let $E_{ti} = E_t \cap (\alpha_i F_{q,r+1})$ for $i = 0, 1, \ldots, 2^{q-r-1} - 1$. For a pair (d_1, d_2) of E_t, $d_1 d_2 \in E_b$ if and only if both d_1 and d_2 are in the same E_{ti} for some i. Thus, we have

$$A_{21}(E) = \sum_{i=0}^{2^{q-r-1}-1} |E_{ti}|(|E_{ti}| - 1)/2 = \sum_{i=0}^{2^{q-r-1}-1} |E_{ti}|^2/2 - n/2.$$

Since $\tilde{D}$ has B-GMC among the designs with $D_t \subset F_{qq}$ and $D_b = H_{r-1} \cup F_{qr}$, $A_{21}(\tilde{D})$ is equal to the minimum value of $A_{21}(D)$ among this class of blocked designs. Thus, the values $|\tilde{D}_{tj}|$ differ at most one with each other for $j = 1, \ldots, 2^{q-r} - 1$. Similarly, $A_{21}(\tilde{E})$ is equal to the minimum value of $A_{21}(E)$ among the designs with $E_t \subset F_{q,q}$ and $E_b = H_r$. So, the values $|\tilde{E}_{ti}|$ also differ at most one with each other for $i = 0, 1, \ldots, 2^{q-r-1} - 1$. Under the condition that

$$\sum_{j=1}^{2^{q-r}-1} |\tilde{D}_{tj}| = \sum_{i=0}^{2^{q-r-1}-1} |\tilde{E}_{ti}| = n,$$

we have

$$\sum_{j=1}^{2^{q-r}-1} |\tilde{D}_{tj}|^2 < \sum_{i=0}^{2^{q-r-1}-1} |\tilde{E}_{ti}|^2,$$

which implies that $A_{21}(\tilde{D}) < A_{21}(\tilde{E})$. The result of (b) follows. □

Appendix B. Supplementary material

Supplementary material, which contains the proof of Theorem 4, can be found online at http://dx.doi.org/10.1016/j.jspi.2015.12.007.

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