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Construction of Orthogonal Nearly Latin Hypercubes

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Orthogonal Latin hypercubes (OLHs) are available for only a limited collection of run sizes. This paper presents a simple algorithm for constructing orthogonal designs that are nearly Latin hypercubes. The algorithm is based on the approach developed by Steinberg and Lin for OLH designs and can generate designs for all run sizes for which a Plackett-Burman design exists. The designs have good univariate projections, although they are not perfectly uniform. They also provide good spatial coverage in higher dimensions. The great gain in sample size flexibility requires just a small sacrifice in the univariate spread of points. Copyright © 2014 John Wiley & Sons, Ltd.

Keywords: algorithm; computer experiments; Plackett–Burman designs

1. Introduction

Latin hypercube designs (LHDs) were proposed by McKay, Beckman, and Conover[2] for exploring properties of a computer simulator subject to uncertainty about the values of its inputs. They are now a popular choice of factorial design for computer experiments.[3–5] An LHD with n runs for a system with k inputs is an $n \times k$ matrix D, each of whose columns is uniformly spaced.

The LHDs, by construction, offer good coverage of each univariate axis. However, there are no guarantees of higher-dimensional properties. Much subsequent research has been devoted to augmenting the basic structure by additional criteria that relate to multivariate properties. In screening settings, when the goal is to identify the most important among a large number of inputs, first-order orthogonality may be a useful property for an LHD. Ye[6] was the first to develop methods for generating orthogonal LHDs (OLHDs). Steinberg and Lin[1] showed the existence of nearly saturated OLHDs, but only for very special choices of n. A number of subsequent papers have added further construction algorithms, but still there is no method for quite general values of n and $k < n$.[7–12]

Table I lists some of the known cases where an n-run OLHD (n even) can be constructed for as many as k factors. See Sun, Liu, and Lin[12] for some additional cases in which n is larger by 1 than some of the run sizes in Table I. Despite the increasing sophistication of the construction methods, the collection of sample sizes is still rather limited, and for many sample sizes, only a small number of factors can be included.

We are convinced that OLHDs will be most useful in factor screening settings, where the goal is to identify the most important among a rather large number of potential factors. The small number of factors for many current constructions is thus a major limitation.

In this paper, we present a simple method for generating first-order orthogonal designs for any value of n with a known Plackett and Burman[13] (hereafter PB) two-level orthogonal design. The designs can accommodate as many as n-4 orthogonal factors, although we recommend limiting the number of factors to at most about $n/2$. The designs have good space-filling properties, at the expense of a slightly non-uniform spread of points on each factor axis. We call them 'orthogonal nearly Latin hypercube designs'. PB designs exist for many multiples of 4 (Hedayat, Sloane and Stufken,[14]) so this greatly extends the realm of application for orthogonal designs in computer experiments.

2. Orthogonal designs that are nearly Latin hypercubes

2.1. The construction algorithm

In this section, we present our construction algorithm. As in our previous work[1] (hereafter SL), the method is based on rotating sets of columns from an orthogonal two-level design D to obtain an orthogonal design with all factors in the range $[-1, 1]$. Here, we apply this idea to obtain designs with much more flexible run sizes than those in SL. Our construction algorithm has the following steps:

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Table I. Known constructions for orthogonal Latin hypercube designs with *n* runs and up to *k* factors

Runs	Maximal factors	Source
16	12	SL[1]
32	16	SLL[12]
48	12	LBS[8]
64	32	LMT[9]
80	12	LBS[8]
96	24	LBS[8]
112	12	LBS[8]
128	48	LBS[8]
144	24	LBS[8]
160	24	LBS[8]
176	12	LBS[8]
192	48	LBS[8]
208	12	LBS[8]
224	24	LBS[8]
240	12	LBS[8]
256	248	SL[1]
512	256	SLL[12]
1024	512	SLL[12]

Sources: Steinberg and Lin (SL[1]); Sun, Liu, and Lin (SLL[12]); Lin, Mukerjee, and Tang (LMT[9]); Lin, Bingham, and Sitter (LBS[8]).

- Step 1: Divide the columns of D into B disjoint sets $D_1, \ldots, D_B$ with t_k columns in set D_k.
- Step 2: For an appropriate $t_k \times t_k$ rotation matrix R_k, map the t_k columns in D_k into t_k new columns by $D_k \mapsto D_k R_k$. The new design is then

$$D_R = [D_1 R_1 \vdots \cdots \vdots D_B R_B] \tag{1}$$

- Step 3: Delete some of the resulting columns if there are more columns than experimental factors. See Section 3.2 for advice on which columns to delete.
- Step 4: The rotation will result in D_R having entries that fall outside $[-1, 1]$. Divide D_R by its largest element (in absolute value) to scale the design back to the unit hypercube.

As R_k is a rotation matrix, $R_k' R_k = I$. Thus, the algorithm generates orthogonal columns, and hence first-order orthogonal designs, whenever the initial design D is orthogonal. As in SL, we use the sequence of rotation matrices that was proposed by Beattie and Lin.[15–17] These matrices rotate columns in sets of size $t = 2^m$ and are defined by the following recursive scheme. Let

$$V_0 = [1] \tag{2}$$

$$V_{m+1} = \begin{bmatrix} V_m & -(2^{2^m})V_m \\ (2^{2^m})V_m & V_m \end{bmatrix} \tag{3}$$

It is easy to check that V_m is orthogonal. The entries of V_m are $1, \pm 2^1, \ldots, \pm 2^{t-1}$, where $t = 2^m$. Each of these integers, with either a plus or minus sign, appears exactly once in each column. Simple re-scaling converts V_m into a rotation matrix,

$$R_m = (1/a_m)V_m \tag{4}$$

with $a_0 = 1$ and $a_m = \left[\prod_{j=1}^{m} \left(1 + 2^{2j}\right)\right]^{1/2}$ for $m = 1, 2, \ldots$. For example,

$$R_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \tag{5}$$

and

$$R_2 = \frac{1}{\sqrt{85}} \begin{bmatrix} 1 & -2 & -4 & 8 \\ 2 & 1 & -8 & -4 \\ 4 & -8 & 1 & -2 \\ 8 & 4 & 2 & 1 \end{bmatrix} \tag{6}$$

SL limited consideration to cases where D was a saturated 2^{p-q} design, and the number of runs $n = 2^{p-q}$ can be written in the form 2^{2^m}. For these special run sizes, SL showed that the columns of the original design matrix could be divided into blocks D_k of $t = 2^m$

columns, with each D_k a single replicate of a full 2^t factorial design. When D_k is a full 2^t factorial, D_kR_m is guaranteed to be a Latin hypercube, and the complete rotated design is an OLHD.

2.2. *Rotated Plackett–Burman designs*

Here, we extend SL by allowing D to be any PB design. The rotated design will still be orthogonal for first-order effects. The number of columns in each set D_k will still be a power of 2. However, from the structure of the PB designs, D_k will typically not be full factorial designs. Rather, D_k will be a subset of a factorial design and might include some replicated rows. Alternatively, it could include all the factorial points but with partial replication. Consequently, the design columns generated by rotating D_k will not have the uniform spread that is required in an LHD.

Let PBn denote the PB design with n runs and n-1 orthogonal columns. We can divide the columns into any collection of disjoint sets whose size is a power of 2. Note that the sets do not have to be all of the same size. Thus, in general, our rotated design can include up to n-4 first-order orthogonal columns.

2.2.1. Example. Consider generating a 40-run design using sets D_k with eight columns. Each set has 256 possible distinct rows, and most of the rows in eight-factor projections of PB40 are distinct. However, it is also common to find at least two repeated rows in eight-factor projections, and these repeat rows will generate repeat rows in D_kR_3 as well. Therefore, the columns of D_kR_3 will have close to 40 distinct values.

The univariate projections of D_kR_3 will not have the uniform distribution required of an LHD because D_k has only some of the rows of the full 2^8 factorial. However, the values obtained will typically have nearly a uniform distribution on $[-1,1]$. Thus, the design will maintain the orthogonality with only a small penalty on achieving perfectly uniform one-factor projections. The quality of the one-factor projections can be seen in Figure 1, which shows Q–Q plots for four factors in a 40-run design rotated in sets of eight. In the next section, we explore the properties of the design and show how they are related to projection properties of the PB design.

We can allocate the 39 columns of PB40 to four sets of eight columns each, with seven columns 'left over'. The resulting rotated design has 32 factors and 40 runs. We could create an additional set using any four of the final seven columns and rotating with R_2. However, the matrix D_k for such a set can have only 16 distinct rows, so the resulting columns of D_kR_2 will have at most 16 distinct entries. The additional four design columns will have considerable replication.

Which matrix R_m should be used to rotate the columns of the PB design? With PB40, options are R_2, R_3, R_4, and R_5, corresponding to groups of 4, 8, 16, or 32 columns, respectively. As we will show in the next section, increasing m (say combining two sets of eight into a single set of 16) leads to only small perturbations in the design points. So it will have a limited effect on most design properties. However, it will reduce (and typically eliminate) the presence of exact repeat values for any rotated factor. Elimination of repeat values is often important in computer experiments, so in general, it is desirable to make m the largest integer for which $2^m < n$.

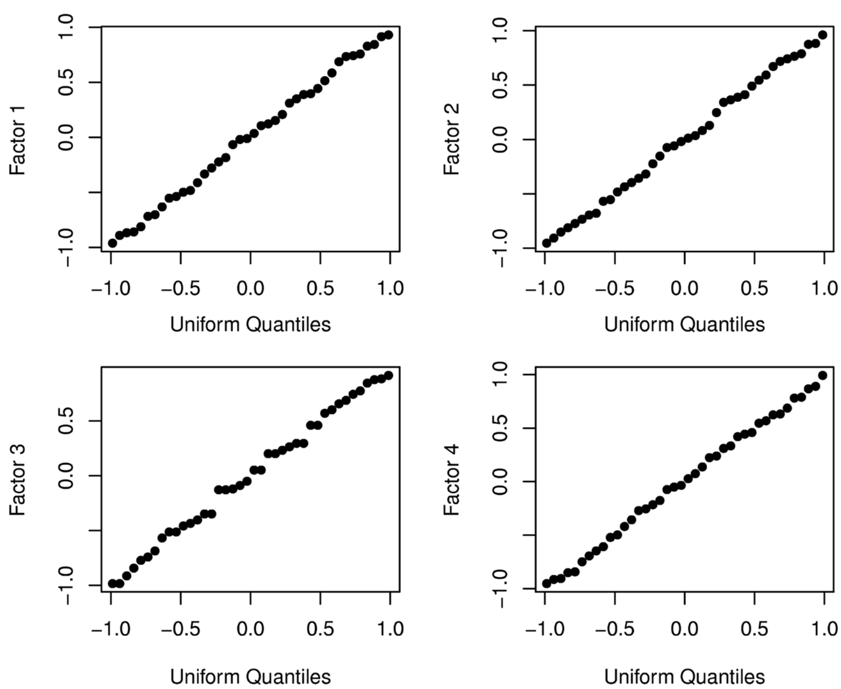

Figure 1. Uniform Q–Q plots for four factors from a rotated 40-run Plackett–Burman design.

Using large values of m may leave many 'unused' columns. The remaining columns could be rotated in smaller sets. For example, with a 60-run design, our recommendation would be to rotate groups of 32 columns, and only one such group could be formed, leaving 27 columns out of the rotated design. From these columns, one could rotate two additional sets, one with 16 columns and another with 8 columns, generating a rotated design with 56 columns.

3. Rotation designs and Plackett–Burman projections

The properties of our rotated PB designs are closely related to the projection properties of the original PB design. That relationship can be exploited to improve the grouping of PB columns into rotation sets and also when selecting which columns from a rotated design to keep and which to drop.

3.1. Univariate properties

Consider the columns formed from the mapping $D_k \mapsto D_k R_m = (1/a_m) D_k V_m$. Denote the jth column of $D_k V_m$ by $D_{V,u}$. This column is a linear combination of the 2^m columns in D_k with weights given by the jth column of V_m, which are the numbers $1, 2, \ldots, 2^{t-1}$, $(t = 2^m)$, possibly multiplied by -1. Sorting the columns of D_k by the corresponding powers of 2, we can write

$$D_{V,j} = \sum_{i=1}^{t} s_i 2^{t-i} c_i \tag{7}$$

where c_i is a column in D_k and s_i is either 1 or -1. Note that the sum is organized so that c_1 is the column that is multiplied by the highest power of 2.

The binary expansion shows that the sign of each entry in $D_{V,u}$ is completely determined by the value of c_1. Similarly, whether the entry is in the first versus second (or third versus fourth) quartile of $D_{V,u}$ is determined by c_2. The orthogonality of the PB design thus guarantees that exactly 1/4 of the entries of $D_{V,u}$ will be in each quartile. In general, if we divide the $D_{V,u}$ axis into 2^r bins of equal width $(r \leq t)$, the number of design points in each bin will be determined by the projection of the PB design on the columns $c_1, c_2, \ldots, c_r$. For example, the first four columns determine how many points will be in each of 16 equal-width bins. For many practical design sizes, this should be sufficient resolution for looking at uniform spread of a rotated design column. Thus, a natural objective is to arrange the PB columns in sets for which each group of four columns has a 'balanced' four-factor projection, with nearly equal numbers of repetitions of each of the 16 factorial points.

In a rotation set with eight factors, there are 256 possible values. So each entry of $D_{V,u}$ is determined to the nearest 2^{-8} of the range of possible values. Using sets of 16 columns rather than eight can affect the resolution only at higher levels, shifting points within these 256 bins, but not moving any points to another bin. The eight columns with the largest entries in the corresponding row of R_4 will dominate the entries in $D_{V,u}$. Hence, our comment in the previous section that using sets of 16 columns with PB40 rather than sets of eight columns will achieve distinct values for the levels of each factor but will leave previously identical levels close together.

3.2. Bivariate properties

The relationship between pairs of rotated columns is also determined by projection properties of the PB design. Consider a second rotated column of $D_k V_m$ and write it as

$$D_{V,j} = \sum_{i=1}^{t} s_i 2^{t-i} d_i \tag{8}$$

Look at the scatterplot of $D_{V,j'}$ versus $D_{V,u}$, divided into four quadrants. The quadrant of each point on the $D_{V,u}$ axis is determined by c_1 and on the $D_{V,j'}$ axis by d_1. By construction, these are different columns of the PB design, so we are assured that exactly 1/4 of the points will fall in each quadrant.

We can divide the plot into 16 squares by using four intervals on each axis. The number of points in each square will be determined by the projection of the PB design onto the columns c_1, c_2, d_1, and d_2. Thus, balanced projections are also desirable to achieve rotated designs with good spatial coverage in each two-factor projection.

The least balanced two-factor projections occur for adjacent columns in the rotated design. From the construction of V_m, each such pair of columns in $D_k V_m$ will have $c_1 = d_2$ and $c_2 = d_1$. Although these two columns are orthogonal, this two-factor projection will have points in only four of the 16 squares. If columns can be dropped from the rotated design, we recommend deleting one column from each of these pairs.

Figure 2 shows scatterplots from a rotated PB40 design. The first two plots show good spatial coverage. The third plot shows factor 1 with factor 4. The column with the highest power of 2 for factor 1 (4) has the fourth highest power of 2 for factor 4 (1). This plot shows some 'clumping' of the points. Factor 1 and factor 2 are adjacent columns, and the points are limited to four squares.

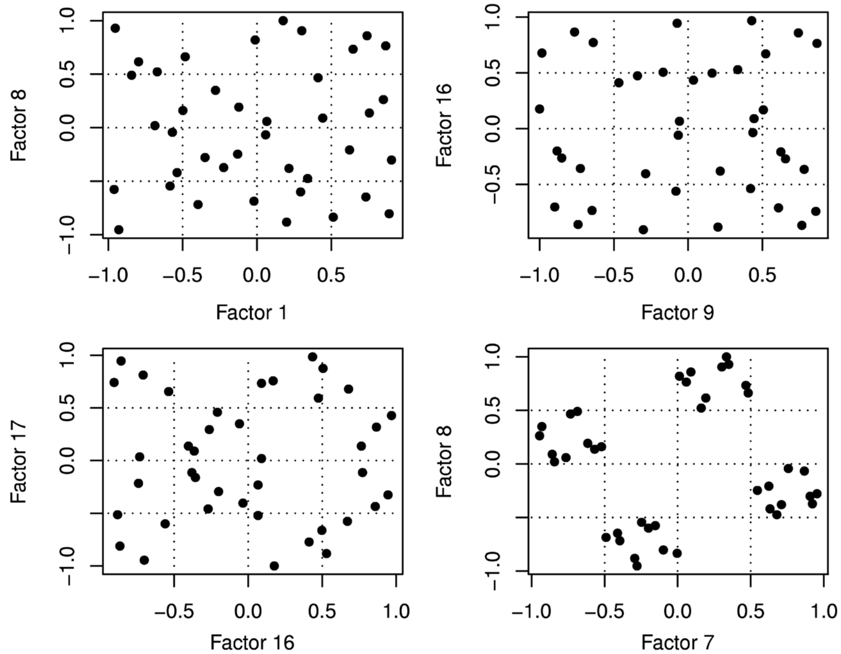

Figure 2. Scatterplots for four pairs of factors from a rotated 40-run Plackett–Burman design.

4. Four-factor projections of Plackett–Burman designs

Section 3 shows that the properties of rotated PB designs depend on the underlying projection properties of the PB design. The spread of values in a rotated column reflects the pattern of points in the three or four PB columns that lead its expansion. The bivariate projection of two rotated columns depends strongly on four PB columns, the two columns that lead the expansion of each of the rotated columns. Thus, properties of four-factor projections of PB designs are relevant to determining the properties of our rotated designs. Moreover, knowledge of four-factor projections could be instrumental in grouping the PB columns prior to rotation so as to create rotated designs with better properties.

Only limited research has been carried out on the projection properties of PB designs and most of that has been limited to three-factor projections. Moreover, the research has focused primarily on whether or not projections include all possible level combinations, without regard to whether each one appears roughly the same number of times. See Cheng[18] and Tyssedal[19] for good summaries.

We have carried out some initial empirical investigation on the properties of four-factor projections of PB designs. In our work, we have used the 40-run, 44-run and 48-run designs corresponding to the Hadamard matrices provided by N. J. A. Sloane on his web site (www.research.att.com/njas/hadamard/index.html). These designs were constructed by the Paley[20] method. We used the 52-run design generated by JMP.[21] We give a brief summary here.

With the 40-run design, there are 835 four-factor projections that are fivefold replications of a 2^{4-1} design with only eight distinct points. A further 1140 projections have only 12 distinct points, five repeated twice, six repeated four times, and one occurring six times. Another 2736 projections have 15 distinct runs, 10 repeated twice and five repeated four times. All the other projections include all 16 points. The most even distribution has two replicates of the 2^4 design together with a single replicate of a 2^{4-1} fraction; this pattern occurs for 36,480 projections. A slightly less balanced projection pattern (two singletons, six each with two and three replicates and two with four replicates) occurs for 31,920 projections.

For the 44-run design, there are only four distinct projection types. Three of the four include all 16 points, and about 93% of the projections fall into one of those three types. The most balanced projection pattern has five points duplicated, 10 points in triplicate, and one point that occurs four times. This design is composed of two replicates of the 2^4 design together with a single replicate of a 12-run PB design. Of the other patterns that include all the points, one has three singletons, and one point that occurs five times, whereas the other has just one singleton and no points that occur more than four times.

The 48-run design has seven projection patterns, four of which account for 88% of all four-factor projections. No projections include three replicates of the 2^4 factorial design. The most balanced projections have eight points replicated three times, four points replicated twice, and four points replicated four times; this pattern occurs for nearly 40% of the projections. These designs are composed of two replicates of the 2^4 design along with 16 runs from one of the irregular Hadamard matrices. These points follow one of two patterns. One possibility is that the points replicated three times form a 2^{4-1} fraction in which a two-factor interaction is aliased; those replicated four times have two runs at each level for each factor, and all factors are orthogonal except the pair that was aliased in the first set. In the second option, the points replicated three times form a 2^3 design with the fourth factor held constant; those replicated four times have the fourth factor at its opposite level with all other factors at two levels and orthogonal to one another. Another common projection pattern has six points, rather than eight, replicated three times, with one point replicated just twice and another five times. In 6% of the projections, one of the 16 factorial points does not occur.

The 52-run design presents a more varied and richer picture. There are nearly 250,000 four-factor projections and 18 distinct projection patterns. More than 95% of the projections include all 16 points. Only 95 projections have 12 or 13 distinct points, and one of these patterns has a single run replicated eight times. The best balance is achieved by 21% of the projections and has five points repeated four times, 10 points repeated three times, and one point repeated twice. The structure of these projections can be described in terms of the single run with two replicates; the five runs with four replicates are the four runs that differ from it for exactly one factor and the run that differs from it for all four factors. All other projections have at least one point with at least five replications.

We have used the aforementioned characterizations to look for good projections sets in rotating PB designs. However, we do not have a formal algorithm that can be implemented. Further work is needed to explore the four-factor projection properties of PB designs and to fully exploit them for creating good rotated PB designs.

5. Rotated foldover designs

Orthogonality properties of factorial designs can be enhanced by the foldover technique, in which an n-run design matrix D is complemented by its mirror image $-D$. If the columns of D are orthogonal to one another, then in the combined design, with $2n$ runs, the columns are also orthogonal to pure quadratic effects and to two-factor interaction columns. As shown in Steinberg and Lin,[1] rotation preserves these additional orthgonalities. One additional factor can be added to the design, which is 1 for all rows in D and -1 for all rows in $-D$.

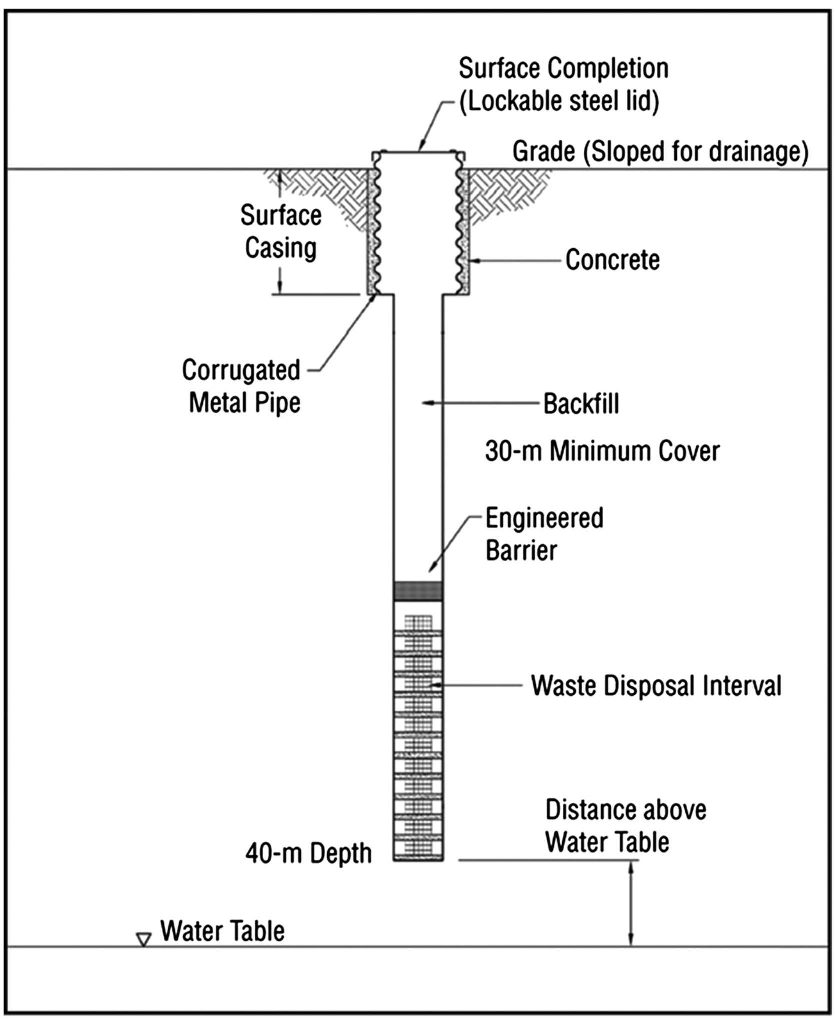

Figure 3. Borehole model illustration.

6. Case study: borehole model

The borehole model has been studied by a number of different approaches: Harper and Gupta,[22] Worley,[23] Morris *et al.*,[24] An and Owen,[25] and Fang and Lin[26]. In this study of flow rate of water from an upper aquifer to a lower aquifer, the aquifers are separated by an impermeable rock layer, but there is a borehole through that layer connecting them. Illustrated in Figure 3 is a model used to describe the flow of water through the borehole from the ground surface. The model is based on assumptions of (i) no groundwater gradient; (ii) steady-state flow from the upper aquifer into the borehole and from the borehole into the lower aquifer; and (iii) laminar, isothermal flow through the borehole. In this model, the response y is the flow rate through the borehole and can be expressed as

$$y = \frac{2\pi T_u[H_u - H_l]}{ln\left(\frac{r}{r_w}\right)\left[1 + \frac{2LT_u}{ln(r/r_w)r_w^2K_w} + \frac{T_u}{T_l}\right]} \tag{9}$$

where the eight-input variables and their ranges are described in Table II.

For illustration and comparison, the case for a design with run size $n = 40$ is presented in this section. The proposed orthogonal nearly LHD (Section 2.2) is displayed in Appendix A. We compared it with two other LHDs—the maximin LHD,[27] which maximizes the minimal distance between any two design points, and the conventional LHD. The maximin LHD is obtained from the R-package via the command 'lhs'. For random LHD, 1000 designs were generated by random permutation and evaluated.

For each design, data were generated from the 'true model'. Then a first-order linear model with eight (8) variables was fitted. Four comparison criteria were then evaluated: R^2, CN = condition number, VIF = the variance inflation factor, and SD = standard deviation of the estimated parameters.

The results are shown in the succeeding text. Note that the random LHDs generate 1000 values for each criterion and are displayed via a density curve or an average. The general observations are as follows:

- The R^2 for the proposed design is superior to the maximin LHD and greater than most random LHDs, as shown in Figure 4.
- The CN for the proposed design is better (lower) than the maximin LHD and almost all random LHDs, as shown in Figure 5.
- The VIFs for the proposed design always equal the optimal value of 1, because of its orthogonality. The VIFs for the maximin LHD range from 1.03 to 1.25.
- The SDs for the proposed design are smaller than for the other designs for all parameters, as shown in Table III.

Table II. Borehole study: the variables, their units, and their ranges

Symbol	Description	Units	Lower limit	Upper limit
r_w	Radius of borehole	m	0.05	0.15
r	Radius of influence	m	100	50,000
T_u	Transmissivity of upper aquifer	m^3/yr	63,070	115,600
T_l	Transmissivity of lower aquifer	m^3/yr	63.1	116
H_u	Potentiometric head of upper aquifer	m	990	1110
H_l	Potentiometric head of lower aquifer	m	700	820
L	Length of borehole	m	1120	1680
K_w	Hydraulic conductivity of borehole	m/yr	9855	12,045

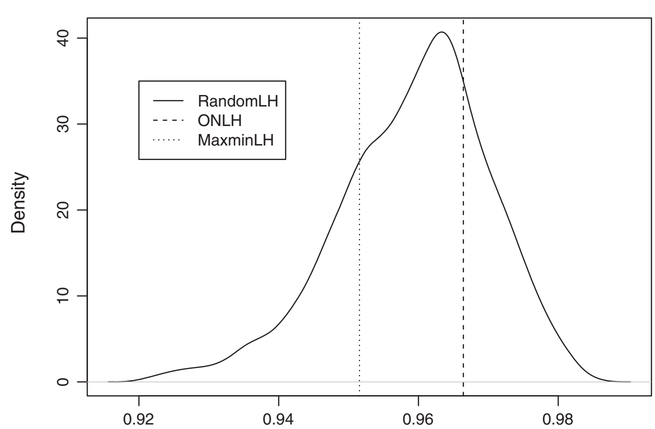

Figure 4. Comparison on $\mathbf{R^2}$ for random Latin hypercube design (LHD), maximin LHD, and orthogonal nearly LHD.

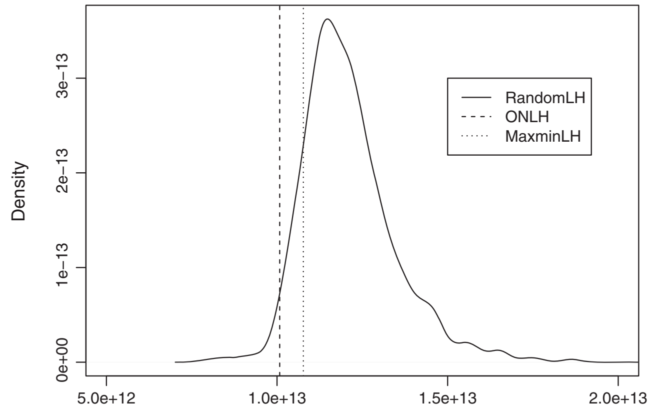

Figure 5. Comparison on condition number for random Latin hypercube design (LHD), maximin LHD, and orthogonal nearly LHD.

Table III. Borehole study: the standard errors for each term in the first-order model

Parameters	Random LHD	Maximin LHD	ONLHD
Intercept	78.8	91.9	63.1
r_w	63.3	69.3	50.7
r	1.26×10^{-4}	1.42×10^{-4}	1.01×10^{-4}
T_u	1.20×10^{-4}	1.32×10^{-4}	0.97×10^{-4}
T_l	0.0527	0.0581	0.0422
H_u	0.119	0.141	0.096
H_l	0.0527	0.0586	0.0422
L	0.0113	0.0130	0.0091
K_w	0.00289	0.00314	0.00231

For the random LHDs, the average standard error is shown.
LHD: Latin hypercube design; ONLHD: orthogonal nearly Latin hypercube design.

Table IV. Borehole study: correctly and falsely identified factors from fitting a first-order model

Added factors	Identified	Random LHD	Maximin LHD	ONLHD
10	Correct	4.8	4	6
10	False	0.54	1	0
20	Correct	3.9	3	6
20	False	0.95	0	3

For the random LHDs, the average numbers are shown.
LHD: Latin hypercube design; ONLHD: orthogonal nearly Latin hypercube design.

In general, the proposed design outperforms the maximin LHD and the random LHDs, on the basis of these four criteria. We also compared the designs with respect to their factor screening ability. To do so, we added factors to the design that have no effect on the outcome. The first set of designs had 10 additional factors and are at the size we advocate, with about $n/2$ factors. The second set of designs included 20 additional factors and so had more factors than we recommend for screening with orthogonal nearly LHD designs. First-order regressions were fitted, and factors were screened on the basis of achieving a p-value of 0.05 or less. We recorded for each design how many of the eight actual factors were identified and how many of the extra factors were falsely identified. The results are shown in Table IV. With 10 additional factors, the orthogonal nearly LHD is clearly the most successful. With 20 additional factors, it continues to identify the most true factors but now also suffers from false identifications. This design cannot avoid having some pairs of factors with two-factor projections concentrated in just four cells of a 16-cell grid.

7. Concluding remarks

Our orthogonal nearly Latin hypercube designs are rotations of PB designs and greatly extend the class of designs proposed by Steinberg and Lin.[1] These designs have great sample size flexibility and can be applied whenever there is a known PB design (almost any sample size n that is a multiple of 4), whereas the original SL designs could be used for only very special sample sizes. Like SL, the rotated PB designs are first-order orthogonal. The designs are not Latin hypercubes, but they achieve nearly uniform spread for each factor. The designs are nearly saturated and can include as many factors as the largest multiple of 4 that is smaller than n. However, we advise using not more than half this number of factors to avoid bivariate projections with poor spatial coverage. The aforementioned recommendation follows our results that show how the quality of the two-factor projections is related to projections of the initial PB design. It is also supported by the screening results of our case study, where including too many factors led to a large number of false identifications.

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Appendix A

The proposed design for borehole study ($n = 40$ and $k = 8$)

Runs	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
1	−0.498	−0.671	0.725	−0.451	0.914	−0.357	−0.522	0.161
2	−0.02	−0.843	−0.788	0.106	−0.694	0.388	−0.49	−0.686
3	0.843	−0.02	−0.106	−0.788	−0.388	−0.694	0.686	−0.49
4	−0.278	−0.953	0.247	0.545	−0.584	0.129	−0.929	0.349
5	0.827	−0.027	−0.043	−0.757	−0.137	−0.569	−0.318	−0.992
6	−0.333	−1	0.2	0.6	0.294	0.882	−0.176	−0.529
7	−0.184	−0.514	−0.263	−0.945	−0.075	−0.851	−0.835	0.004
8	−0.482	−0.663	0.663	−0.482	0.663	−0.482	0.482	0.663
9	−0.412	−0.647	0.647	−0.412	−0.467	−0.733	0.733	−0.467
10	0.514	−0.184	0.945	−0.263	0.851	−0.075	−0.004	−0.835
11	0.686	−0.49	0.388	0.694	0.106	0.788	0.843	−0.02
12	−0.067	−0.867	−0.867	0.067	0.059	0.765	0.765	−0.059

(*Continues*)

Appendix A. *(Continued)*

Runs	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
13	0.757	−0.043	−0.027	−0.827	0.992	−0.318	−0.569	0.137
14	0.733	−0.467	0.467	0.733	−0.647	0.412	−0.412	−0.647
15	−0.012	−0.82	−0.82	0.012	−0.82	0.012	0.012	0.82
16	−0.224	−0.553	−0.373	−0.922	0.553	−0.224	0.922	−0.373
17	0.584	−0.129	0.929	−0.349	−0.278	−0.953	0.247	0.545
18	0.741	−0.443	0.435	0.639	−0.773	0.035	0.09	0.859
19	0.914	−0.357	−0.522	0.161	0.498	0.671	−0.725	0.451
20	0.929	−0.349	−0.584	0.129	0.247	0.545	0.278	0.953
21	0.122	0.796	0.78	−0.051	−0.937	0.365	0.616	−0.192
22	0.396	0.718	−0.718	0.396	0.718	−0.396	0.396	0.718
23	−0.718	0.396	−0.396	−0.718	0.396	0.718	−0.718	0.396
24	0.153	0.576	0.255	0.961	0.576	−0.153	0.961	−0.255
25	−0.702	0.404	−0.459	−0.749	0.145	0.592	0.286	0.898
26	0.208	0.624	0.302	0.906	−0.302	−0.906	0.208	0.624
27	0.31	0.89	−0.239	−0.561	0.082	0.875	0.804	−0.098
28	0.106	0.788	0.843	−0.02	−0.686	0.49	−0.388	−0.694
29	0.035	0.773	0.859	−0.09	0.443	0.741	−0.639	0.435
30	−0.89	0.31	0.561	−0.239	−0.875	0.082	0.098	0.804
31	−0.812	0.114	0.114	0.812	−0.114	−0.812	−0.812	0.114
32	0.443	0.741	−0.639	0.435	−0.035	−0.773	−0.859	0.09
33	−0.631	0.42	−0.475	−0.678	−0.984	0.341	0.537	−0.231
34	−0.859	0.09	0.035	0.773	0.639	−0.435	0.443	0.741
35	0.388	0.694	−0.686	0.49	0.843	−0.02	−0.106	−0.788
36	0.349	0.929	−0.129	−0.584	−0.545	0.247	−0.953	0.278
37	−0.961	0.255	0.576	−0.153	0.255	0.961	−0.153	−0.576
38	−0.867	0.067	0.067	0.867	0.765	−0.059	−0.059	−0.765
39	−0.537	0.231	−0.984	0.341	−0.475	−0.678	0.631	−0.42
40	−0.553	0.224	−0.922	0.373	−0.224	−0.553	−0.373	−0.922

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David Steinberg is Professor of Statistics in the Department of Statistics and Operations Research at Tel Aviv University. Hehas published more than 100 papers in refereed journals along with several book chapters and about 15 papers in conference proceedings. His field of research specialization is the statistical design of experiments, including factorial experiments, Latin hypercubes, computer experiments, robust parameter design experiments and seismic networks. He has worked on numerous applications in a variety of fields. Steinberg received theGeorge Box Medal from ENBIS in 2013. From 2008–2010 he served as Editor of Technometrics (and as Editor–Elect during 2007). He is currently on the editorial boards of the Journal of Uncertainty Quantification and Quality Technology and Quantitative Management. He was Section Editor for Experimental Design for the Encyclopedia of Statistics in Quality and Reliability.

Dr. Dennis Lin is a University Distinguished Professor of Statistics and Supply Chain Management at Penn State University. His research interests are quality engineering, industrial statistics, data mining and response surface.He has published near 200 papers in a wide variety of journals. Dr. Lin is an elected fellow of ASA, IMS and ASQ, an elected member of ISI, a lifetime member of ICSA, and a fellow of RSS. He serves or has served asa co-editor for ASMBIand an associate editor for various (more than 10) journals.He is an honorary chair professor for various universities, includinga Chang-Jiang Scholar of China at Renmin University, National Chengchi University (Taiwan) and Fudan University. He is the recipient of the 2004 Faculty Scholar Medal Award at Penn State University. His recent awards include Don Owen Award (ASA), Youden Address (ASQ) and Loutit Lecturer (SSC).